RIEMANN MATRICES FOR HYPERELLIPTIC SURFACES WITH INVOLUTIONS OTHER THAN THE INTERCHANGE OF SHEETS

John Schiller

Let $\omega_1, \cdots, \omega_g$ form a basis for the holomorphic differentials on a compact Riemann surface S of genus g, and let (a_i, b_i) $(i = 1, \cdots, g)$ form a set of retrosections (one-cycle representatives of a homology basis for S, where $\delta(a_i, b_j) = \delta_{ij}$, $\delta(a_i, a_j) = 0 = \delta(b_i, b_j)$, δ being the bilinear, skew-symmetric intersection number); then the $g \times 2g$ matrix

$$(A \;\; B) \equiv \left(\left(\int_{a_j} \omega_i \right) \left(\int_{b_j} \omega_i \right) \right)$$

is called a *period matrix* for S. By a change of basis for the holomorphic differentials, the matrix A can be reduced to the multiplicative identity (the new basis is said to be *normalized* with respect to (a_i, b_i)), and then B becomes $A^{-1}B$, which is symmetric with positive-definite imaginary part and is called the *Riemann matrix for* S *with respect to* (a_i, b_i). Torelli's theorem says that if a surface S has the same Riemann matrix with respect to (a_i, b_i) as a surface S' has with respect to (a_i', b_i'), then there exists a conformal homeomorphism from S onto S' taking either a_i to a_i' and b_i to b_i' or a_i to $-a_i'$ and b_i to $-b_i'$ (see [5, pp. 27-28] and the references cited there). If S' (and therefore S) is hyperelliptic, then conformality of one map implies conformality of the other, since the two maps then differ by the "interchange of sheets" on S', which is conformal. Every conformal equivalence class of hyperelliptic surfaces of genus g that have involutions (conformal self-homeomorphisms of order 2) other than the interchange of sheets contains a surface whose equation is $w^2 = f(z^2)$, where f(x) is a complex polynomial of degree $g + 1$. This is a particular case of a result due to Hurwitz [2, p. 257]. The purpose of this note is to show that such surfaces can be classified according to their Riemann matrices.

If a surface S has the equation $w^2 = f(z^2)$, then in addition to the interchange of sheets $\iota: (z, w) \to (z, -w)$, the surface has at least two involutions, namely

$$\tilde{\sigma}: (z, w) \to (-z, w) \quad \text{and} \quad \hat{\sigma} \equiv \iota\tilde{\sigma}: (z, w) \to (-z, -w).$$

The natural projection $\tilde{\pi}$ from S to the quotient surface $\tilde{S} \equiv S/(1, \tilde{\sigma})$ is given concretely by $(z, w) \to (z^2, 2w) \equiv (\tilde{z}, \tilde{w})$, from which we see that $\tilde{S}$ has the equation $\tilde{w}^2 = 4f(\tilde{z})$ and that the differentials $(\tilde{z}^i/\tilde{w})d\tilde{z}$ $(i = 0, 1, \cdots)$ on $\tilde{S}$ lift to the "odd" differentials $(z^{2i+1}/w)dz$ on S. Similarly, the projection $\hat{\pi}$ from S to the quotient surface $\hat{S} \equiv S/(1, \hat{\sigma})$ is given by $(z, w) \to (z^2, 2zw) \equiv (\hat{z}, \hat{w})$, $\hat{S}$ has equation $\hat{w}^2 = 4\hat{z}f(\hat{z})$, and the differentials $(\hat{z}^i/\hat{w})d\hat{z}$ on $\hat{S}$ lift to the "even" differentials $(z^{2i}/w)dz$ on S. Note that if g is even, then both $\tilde{S}$ and $\hat{S}$ are of genus g/2, whereas if g is odd, then $\tilde{S}$ is of genus $(g - 1)/2$ and $\hat{S}$ is of genus $(g + 1)/2$. In either case, we can construct a model for S by pasting together two slit copies of $\tilde{S}$ or $\hat{S}$, and then ι, $\tilde{\sigma}$, $\hat{\sigma}$ appear as rotations through 180°. (See Figure 1 for $g = 4$, which is typical for even genus, and Figure 2 for $g = 5$, which is typical for odd genus.) It

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