ON THE COEFFICIENTS OF FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

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1. INTRODUCTION

In this note, we discuss the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the unit disc U, satisfy the condition $f'(z) \neq 0$ in U, and map U onto a domain with bounded boundary rotation (for a definition of this concept, see [3]). In particular, we denote by V_k the family of functions that satisfy the above conditions and map U onto a domain with boundary rotation at most $k\pi$. V. Paatero [3] showed that $f \in V_k$ if and only if

$$f(z) = \int_0^z \exp\left\{\int_0^{2\pi} \log(1 - ze^{-it})^{-1} d\mu(t)\right\} dz, \tag{1.1}$$

where $\mu(t)$ is real-valued and of bounded variation on $[0, 2\pi]$ and satisfies the conditions

$$\text{i) } \int_0^{2\pi} d\mu(t) = 2, \qquad \text{ii) } \int_0^{2\pi} |d\mu(t)| \leq k.$$

V_2 is precisely the class of normalized univalent functions that map U onto a convex domain, and it is known [3] that for $2 \leq k \leq 4$, V_k consists only of univalent functions.

In spite of considerable effort, the problem of determining

$$A_n(k) = \max_{f \in V_k} |a_n| \tag{1.2}$$

remains unsolved, except for $k = 2$ and $k = 4$.

K. Loewner [2] proved that $A_n(2) = 1$, and A. Rényi [5] proved that $A_n(4) = n$. Rényi's result shows that

$$A_n(k) \leq n \qquad (k \leq 4); \tag{1.3}$$

in addition Rényi proved that

$$A_n(k) \leq n^{k-2}, \tag{1.4}$$

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