EICHLER INTEGRALS AND THE AREA THEOREM OF BERS

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1. Let Γ be a group of fractional linear transformations acting on the extended complex plane. We denote the limit point set by Λ and the set of discontinuity by Ω. It is assumed that Λ is infinite, and that Ω is not empty.

The orbit space $S = \Omega/\Gamma$ falls into components S_i that inherit the complex structure of the plane. On each component there is an invariant Poincaré metric $ds = \lambda\,|dz|$ with curvature -1.

L. Bers [2] has recently proved the following remarkable fact:

THEOREM (Bers). *If* Γ *can be generated by* N *elements, the total Poincaré area of* S *is at most* $4\pi(N - 1)$.

In this paper we give a different version of Bers' proof. It is based on the same idea, but it uses singular Eichler integrals rather than Beltrami differentials.

Sections 2 to 9 are restatements of known facts in the form that we need. The proof is in Sections 10 to 14, and in Section 15 we show that the number of components is at most $18(N - 1)$, a slight improvement on the bound given by Bers.

2. The projection map $\pi\colon \Omega \to S$ defines a ramification number $n(p) \geq 1$ at every $p \in S$, and the points with $n(p) > 1$ are isolated. They are projections of elliptic fixed points. If the fixed point is placed at 0, for convenience, the projection may be expressed through $\tilde{z} = z^n$, where $n = n(p)$ and $\tilde{z}$ is the value at $\pi(z)$ of a local parameter.

For finitely generated Γ, it had been shown that there are only a finite number of points with $n(p) > 1$ on each S_i. Moreover, S_i can be extended to a compact surface $\overline{S}_i$ by the addition of a finite number of points, and we set $n(p) = \infty$ when $p \in \overline{S}_i - S_i$. In a typical case, the projection near such a point becomes $\tilde{z} = e^{-1/z}$. To a small disk $\tilde{\Delta}$: $(|\tilde{z}| < \delta)$ there corresponds a disk Δ in the z-plane whose center lies on the positive real axis and whose circumference passes through 0. The disk Δ is contained in Ω, and it is mapped upon itself by a parabolic transformation in Γ with fixed point at the origin; all other images of Δ under Γ are disjoint.

The genus of $\overline{S}_i$ is denoted by g_i. We recall that the Poincaré area of S_i is given by

$$I(S_i) = 2\pi\left[\,2g_i - 2 + \sum_{p\,\in \overline{S}_i}\left(1 - \frac{1}{n(p)}\right)\right]. \tag{1}$$

The information given above is contained in [1], and it will be our starting point, as it was for Bers. In other respects, we shall strive to make the presentation self-contained.

3. Let q be any integer. If a meromorphic function ϕ on Ω satisfies $\phi(Az)A'(z)^q = \phi(z)$ for all $A \in \Gamma$, it determines through projection a differential $\tilde{\phi}\,d\tilde{z}^q$ on S.

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