SMOOTHNESS OF APPROXIMATION

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INTRODUCTION

Let K be a Chebyshev subset of a Banach space $(X, \| \ \|)$. Then, by definition, K is closed and for every $x \in X$ there exists a unique $P_K(x) \in K$ satisfying the condition

$$\|x - P_K(x)\| = \inf \{ \|x - k\| : k \in K \} = \text{dist}(x, K).$$

This map P_K from X to K is called the *best approximation operator* (BAO) *supported by* K; its value $P_K(x)$ at x is the *best approximation to* x *out of* K. It is easy to see that P_K is always a *closed projection*, in the sense that $P_K(x) = x$ whenever $x \in K$, and, if $x_n \to x$ and $P_K(x_n) \to y$, then $y = P_K(x)$. We are interested in the following general question: assuming that K is convex, how does $P_K(x)$ vary as a function of x?

For every closed convex subset of X to be a Chebyshev set, it is necessary and sufficient that X be reflexive and strictly convex (the deep part of this result is due to James; see the remarks in [18, Section 4]). We conjecture that these conditions are also sufficient to guarantee that P_K is always continuous. However, P_K need not be continuous whenever K is a linear Chebyshev subspace of an arbitrary Banach space. This will be shown by Example 4, in which K is a subspace of codimension 2. The weakest condition that is known (to the authors) to imply that P_K is continuous is that K be *approximatively compact*, in other words, that every minimizing sequence in K be compact [5]. In particular, every closed convex subset of a uniformly convex space has this property [6].

Very smooth BAO's are characteristic of Hilbert space. Indeed, a Banach space X of dimension greater than 2 has each of the following three properties if and only if it is a Hilbert space: (a) whenever K is a closed subspace of X, it is a Chebyshev set and P_K is a linear operator ([8]; a stronger result has been established in [19]); (b) whenever K is a closed convex subset of X, it is a Chebyshev set and P_K satisfies the Lipschitz condition $\|P_K(x) - P_K(y)\| \leq \|x - y\|$ [2], [17]; (c) whenever K is a 1-dimensional subspace of X, it is a Chebyshev set and P_K is continuously Gateaux differentiable (Theorem 3). (The case where $\dim X \leq 2$ is special because every Chevyshev subspace of codimension 1 supports a linear BAO (Theorem 3).) Thus a major part of this paper will be devoted to kinds of smooth behavior of P_K that are intermediate between continuity and the uniform smoothness that characterizes Hilbert space. There are two ways in which we can weaken the characteristic properties of Hilbert space: by requiring a weaker property to hold for all closed subspaces or convex sets, or by requiring a strong property to hold, but not uniformly.

An example of the first kind is the *uniform Lipschitz* property of approximation. A reflexive and strictly convex space X has property (UL) if there exists a constant

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Received November 3, 1966.
This research was partially supported by the National Science Foundation. Some of the results were originally established in the dissertation of R. Holmes, written at M.I.T. in 1964 under the direction of Professor W. G. Strang.