PAIRS OF REAL 2-BY-2 MATRICES THAT GENERATE FREE PRODUCTS

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1. INTRODUCTION

This note arose from the observation that certain results of Newman [4] follow very simply from a theorem of Macbeath [3]. A result of Brenner [1] follows by a similar argument.

We are concerned with the question when two real unimodular 2-by-2 matrices A and B generate a free group $\mathfrak{G}$, or a group $\mathfrak{G}$ that is the free product of the cyclic group $\mathfrak{A}$ generated by A and the cyclic group $\mathfrak{B}$ generated by B. We obtain several sufficient conditions for $\mathfrak{G}$ to be a free product. The conditions are stated in terms of the arrangement of the fixed points of A, B, AB, and $ABA^{-1}B^{-1}$ under the action of these matrices as linear fractional transformations on the extended real axis.

2. A LEMMA ON PERMUTATION GROUPS

We could obtain our results by applying Macbeath's theorem directly to the action of our matrices as linear fractional transformations acting on the open upper half of the complex plane. In the cases that we treat, this would also enable us to show that $\mathfrak{G}$ operates discontinuously on the upper half-plane and is therefore discrete. But we have preferred to recast Macbeath's theorem in the form of a lemma that enables us to confine attention to the action of $\mathfrak{G}$ on the extended real axis.

LEMMA. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be groups of permutations of a set* Ω, *and let* $\mathfrak{G}$ *be the group generated by* $\mathfrak{A}$ *and* $\mathfrak{B}$ *together. Suppose that* Ω *contains two disjoint nonempty sets* Γ *and* Δ *such that each nontrivial element of* $\mathfrak{A}$ *maps* Γ *into* Δ *and each nontrivial element of* $\mathfrak{B}$ *maps* Δ *into* Γ. *Then either* $\mathfrak{G}$ *is the free product of its subgroups* $\mathfrak{A}$ *and* $\mathfrak{B}$, *or else both* $\mathfrak{A}$ *and* $\mathfrak{B}$ *have order* 2 *and* $\mathfrak{G}$ *is a dihedral group.*

Proof. Suppose that $\mathfrak{A}$ has order greater than 2, hence more than one nontrivial element. Since the images ΓA of Γ under the nontrivial elements A of $\mathfrak{A}$ are disjoint nonempty subsets of Δ, it follows that each such ΓA is properly contained in Δ, that is, $\Gamma A < \Delta$. Let $W = A_1 B_1 \cdots A_n B_n$, where $n \geq 1$, and where $1 \neq A_i \in \mathfrak{A}$ and $1 \neq B_i \in \mathfrak{B}$ for all i. Then $\Gamma A_1 < \Delta$, whence $\Gamma A_1 B_1 < \Delta B_1 \leq \Gamma$; by a continuation of this argument, $\Gamma W < \Gamma$. Thus $W \neq 1$. This shows that $\mathfrak{G}$ is the free product of $\mathfrak{A}$ and $\mathfrak{B}$.

The same conclusion holds if $\mathfrak{B}$ has order greater than 2, and also (trivially) if either $\mathfrak{A}$ or $\mathfrak{B}$ has order 1. The case remains where $\mathfrak{A}$ is generated by an element A, and $\mathfrak{B}$ by B, with $A^2 = B^2 = 1$. Any further relations between A and B can be reduced to the form $(AB)^n = 1$ $(n > 0)$, and indeed to at most one such relation. If such a relation holds, $\mathfrak{G}$ is a dihedral group of order 2n. If none holds, $\mathfrak{G}$ is the infinite dihedral group, a free product of $\mathfrak{A}$ and $\mathfrak{B}$.

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