PAIRS OF MATRICES GENERATING DISCRETE FREE GROUPS AND FREE PRODUCTS

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The purpose of this note is to prove that certain pairs of real 2×2 matrices of determinant 1 generate discrete free groups, and to indicate extensions to pairs of real linear fractional transformations generating discrete free products. The conditions are formulated in terms of the signs of the elements of the matrices, and they may be regarded as a generalization of the situation that exists in the classical modular group Γ and the Hecke groups (see [4], [5], [9], and [10]). Some work along these lines has been done by various authors (see [1], [2], [3], [7], [8], and [11]), but the conditions previously imposed were of a different type, and there is very little intersection with the present work. In addition, it is worth noticing that $\Gamma(2)$ and Γ', the only free normal 2-generator subgroups of Γ, are not covered by the present discussion. Reasonably simple conditions for deciding when an arbitrary pair of elements of SL(2, R) (where R denotes the real field) generates a free group are probably not to be found, and partial answers of the type given here may be the most that can be expected.

Let $G = \{A, B\}$ denote the group generated by the elements A and B of SL(2, R). Then each element W of G has the form

$$W = A^{r_1} B^{s_1} \cdots A^{r_n} B^{s_n}, \tag{1}$$

where the exponents are different from 0 except possibly for r_1 and s_n. If all the exponents are different from 0, we say that W is *of type* (AB). A simple argument shows that

(a) G is free and freely generated by A and B if and only if A and B are not of finite period and no word of type (AB) with $n > 0$ represents the identity,

(b) G is discrete if and only if there is no convergent infinite sequence $W_1, W_2, \cdots$ of distinct words W_i of type (AB).

Our method will consist of deriving inequalities for the elements of the matrices $A^r B^s$ ($rs \neq 0$). The inequalities carry over on multiplication, and they imply the desired results.

THEOREM 1. *Let* A, B *be elements of* SL(2, R). *Suppose that*

$$A = \begin{pmatrix} -a & b \\ -c & d \end{pmatrix}, \quad B = \begin{pmatrix} -\alpha & -\beta \\ \gamma & \delta \end{pmatrix},$$

where $a, b, c, d, \alpha, \beta, \gamma, \delta \geq 0$ *and* $t = d - a \geq 2$, $\tau = \delta - \alpha \geq 2$. *Then the group* $G = \{A, B\}$ *is a free discrete subgroup of* SL(2, R) *and is freely generated by* A, B.

Before embarking on the proof, we should make the following observation: Let us regard A, B as elements of LF(2, R), and add the further restriction that

Received July 10, 1967.