A PROOF OF A STATEMENT OF BANACH ABOUT THE WEAK* TOPOLOGY

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Let B be a Banach space, and let Γ be a linear manifold in the dual space B^*. Let Γ^1 be the manifold consisting of all the points in B^* that are weak* limits of sequences in Γ. By induction, for every ordinal number ξ we define Γ^ξ as follows (with $\Gamma^0 = \Gamma$):

$$\Gamma^\xi = \Big(\bigcup_{\sigma < \xi} \Gamma^\sigma \Big)^1 .$$

Then $\Gamma \subset \Gamma^1 \subset \Gamma^2 \subset \cdots$, and if ξ has a predecessor, then $\Gamma^\xi = (\Gamma^{\xi-1})^1$. If B is separable, there exists a first countable ordinal ξ_0 such that Γ^{ξ_0} is the weak* closure of Γ; ξ_0 is called the *order of* Γ. Banach, in his discussion of this [1, pp. 208-213], proves that for every positive integer n there exists a linear manifold in ℓ^1 of order n. He then states, but does not prove, that there exist linear manifolds in ℓ^1 of arbitrarily high countable orders. He refers to a paper at this point, but the paper never appeared. The corresponding statement for the space H^∞ has been proved by Sarason [6], [7]. In this paper we shall prove the following.

THEOREM. *If ξ is a countable ordinal, there exists an ideal in ℓ^1 of order ξ.*

Let c_0 denote the Banach space of all the complex-valued functions on the integer group that vanish at infinity, with the supremum norm. Then $\ell^1 = (c_0)^*$; let $\ell^\infty = (\ell^1)^* = (c_0)^{**}$. Each of the Banach spaces c_0, ℓ^1, ℓ^∞ can be realized as a space of distributions on the circle group (considered as the real numbers modulo 2π), by the correspondence

$$\{\hat{S}(n): -\infty < n < \infty\} \longleftrightarrow \left\{ S(x) \sim \sum_{n=-\infty}^{\infty} \hat{S}(n)\, e^{inx}: 0 \leq x < 2\pi \right\}.$$

Corresponding to c_0, ℓ^1, ℓ^∞, respectively, are the space PF of *pseudofunctions;* the space W of functions with absolutely convergent Fourier series; and the space PM of *pseudomeasures* (see [3, Appendices I to III]).

Under convolution, ℓ^1 is a group algebra; and W, under pointwise multiplication, is its Gel'fand representation. When we refer to a topology in W, we mean the norm topology unless we say otherwise. If I is an ideal (not necessarily closed) in $W \cong \ell^1$, its *hull* is the closed set

$$h(I) = \{x: f(x) = 0 \text{ for every } f \in I\}.$$

The hull h(I) is empty if and only if I = W. If E is a closed set, then the maximal ideal whose hull is E is the closed ideal $I(E) = \{f \in W: f^{-1}(0) \supset E\}$. The minimal ideal whose hull is E is

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