A UNIQUE-EMBEDDING THEOREM IN CODIMENSION 1

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1. INTRODUCTION

The problem of determining the equivalence classes of embeddings of one manifold into another has been successfully attacked in two general cases. In the first case one assumes that the difference in the dimensions of the manifolds is sufficiently large, and one shows that there exists a unique class of embeddings (see for example [5], [12], or [13]). In the other case one singles out a particular class (generally called unknotted) and reduces its determination to some homotopy problem (see [1], [6], [9], or [12]). There are scattered results of another nature, where one has a unique embedding class even for small codimension (see [2], [3], [4], [7]). This paper demonstrates yet another of the "scattered" results, namely:

THEOREM. *Let* M *be a* 2-*sphere with an odd number of crosscaps. Let* $f: M \to M \times [0, 1]$ *be an embedding into the interior of* $M \times [0, 1]$. *Then* f(M) *can be moved onto* $M \times \{1/2\}$ *by an ambient isotopy that leaves* $M \times \{0, 1\}$ *pointwise fixed.*

If M is any other closed connected 2-manifold, the above result is false. For example, a torus T can be embedded in $T \times [0, 1]$ as the boundary of a tube around a knot, and this can be done in infinitely many different ways.

2. DEFINITIONS AND A LEMMA

We work entirely in the combinatorial category; in fact, we work only with compact combinatorial manifolds of dimension 2 or 3, with (possibly empty) boundary. Thus an embedding f of a manifold M into a manifold N will be piecewise linear. It is said to be *proper* if it carries the interior of M into the interior of N and the boundary of M into the boundary of N (that is, if $f(\partial M) = (\partial N) \cap f(M)$).

Two embeddings f and g of M into N are said to be *ambient isotopic* if there exists a continuous family F_t $(0 \leq t \leq 1)$ of homeomorphisms of N onto itself such that F_0 is the identity map and $F_1(f(M)) = g(M)$. (Thus ambient isotopy is a condition on the images, rather than on the functions themselves.)

Recall, finally, that a compact connected 2-manifold is a sphere with a number m of holes and a number n of handles if it is orientable. If it is not orientable, it has a number m of holes and a number n of crosscaps. The pair (m, n) and the orientability specify the manifold up to homeomorphism (see, for example, [11]).

LEMMA. *Let* M *be a compact, connected* 2-*manifold with vacuous boundary. Let* $f: M \to M \times [0, 1]$ *be a proper embedding. Then either* f(M) *separates* $M \times \{0\}$ *from* $M \times \{1\}$ *in* $M \times I$, *or* f(M) *is the boundary of a* 3-*dimensional submanifold of* $M \times I$.

Proof. Triangulate M and $M \times [0, 1]$ so that f is a simplicial homeomorphism. Now consider the induced homomorphism $f_*: H_2(M; Z_2) \to H_2(M \times [0, 1]; Z_2)$. Since

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