UNKNOTTING POLYHEDRAL HOMOLOGY MANIFOLDS

C. H. Edwards, Jr.

1. INTRODUCTION

Gugenheim [1] showed in 1953 that an n-dimensional polyhedron unknots piecewise linearly in Euclidean k-space E^k if $k \geq 2n + 2$. For piecewise linear manifolds, this result can be improved upon by one dimension—Zeeman's unknotting theorem [11, Theorem 24] includes the fact that every connected, closed, piecewise linear n-manifold ($n \geq 2$) unknots piecewise linearly in E^{2n+1}, as well as the fact that every 1-connected, closed, piecewise linear n-manifold ($n \geq 3$) unknots in E^{2n}. The principal object of the present paper is to establish results of this sort for a larger class of polyhedra that includes all polyhedral homology manifolds, and hence all triangulated, closed, topological manifolds.

We say that the polyhedron X *strongly unknots* in E^k if, given two imbeddings f and g of X into E^k which agree on a subpolyhedron Y of X, there exists an ambient isotopy of E^k which transforms f into g, while leaving pointwise-fixed the image of Y. We prove that an n-dimensional polyhedron X ($n \geq 2$) strongly unknots in E^{2n+1} if $H^n(X - p) = 0$ for each point $p \in X$ (Theorem 3). It follows that every compact, connected polyhedral homology n-manifold ($n \geq 2$) strongly unknots in E^{2n+1}. This result is then used to prove that if M is either a connected, orientable polyhedral homology n-manifold ($n \neq 2$) with $H_1(M) = 0$, or a compact triangulated topological n-manifold ($n \neq 2$) with nonempty boundary, then M unknots in E^{2n} (Theorem 5 and Corollary 5, respectively).

The proofs of these results make use of Zeeman's unknotting theorem, and they hinge upon the following question. If the n-dimensional polyhedron X collapses to the subpolyhedron Y, and Y unknots in E^k, is it true that X also unknots in E^k? It is always true if $k > 2n$ (Lemma 2), but it is generally false if $k \leq 2n$. However, in the critical case $k = 2n$, it is true provided that X satisfies a certain local unknotting condition (Theorem 4). The proof of Theorem 4 is based on the work of Lickorish [3] on the piecewise linear unknotting of cones.

2. DEFINITIONS AND BASIC FACTS

The subset X of a Euclidean space E^n is called a *polyhedron* if there exists a finite simplicial complex K in E^n such that $|K| = X$. The complex K is then called a *triangulation* of X. The map f of the polyhedron X into a Euclidean space is *piecewise linear* if the triangulation K can be chosen so that f is linear on each simplex of K.

A *piecewise linear set* is a subset Y of a Euclidean space such that each point of Y has a neighborhood (in Y) whose closure is a polyhedron. A map of the piecewise linear set Y into a Euclidean space is *piecewise linear* if its restriction to each subpolyhedron of Y is piecewise linear. *Throughout this paper, we shall work within the category of piecewise linear sets and maps.*

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