ON RECURSIVE SETS AND REGRESSIVE ISOLS

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1. INTRODUCTION

We shall assume that the reader is familiar with the concepts and main results of the papers listed as references. We let E denote the collection of all nonnegative integers *(numbers)*, Λ the collection of all isols, Λ^* the collection of all isolic integers, and Λ_R the collection of all regressive isols. It is known that $E \subseteq \Lambda_R \subseteq \Lambda$ and that each of the collections $\Lambda_R - E$ and $\Lambda - \Lambda_R$ has the cardinality of the continuum. In [7] and [8], A. Nerode associated with every recursive function $f: E \to E$ a function $D_f: \Lambda \to \Lambda^*$, and with every recursive set of numbers α a set α_Λ of isols. D_f is an extension of f from E to Λ, and $\alpha \subseteq \alpha_\Lambda$. In [9], Nerode proved the following result: let f be a recursive and eventually combinatorial function; then $D_f(\Lambda) \subseteq (f(E))_\Lambda$, and $D_f(\Lambda) = (f(E))_\Lambda$ if and only if there exists a number n such that $f(n), f(n+1), \cdots$ is an arithmetic progression. This result motivated the problem considered in this paper. For a recursive set α, we define $\alpha_R = \Lambda_R \cap \alpha_\Lambda$. We are interested in comparing the two collections $D_f(\Lambda_R)$ and $(f(E))_R$ in the case where f is an eventually increasing recursive function.

A function $f: E \to E$ is *increasing* if $x < y$ implies $f(x) \leq f(y)$, and *eventually increasing* if there exists a number n such that the function $g(x) = f(x + n)$ is increasing. It was proved in [1] that if f is a recursive and eventually increasing function, then $D_f: \Lambda_R \to \Lambda_R$. The main result of this paper states that if f is a recursive and eventually increasing function, then $D_f(\Lambda_R) = (f(E))_R$.

2. EXTENSIONS

Let α be a set of numbers. The *characteristic* function of α, denoted by c_α, is defined by

$$c_\alpha(x) = \begin{cases} 0 & \text{if } x \in \alpha, \\ 1 & \text{if } x \notin \alpha. \end{cases}$$

If α is a recursive set, then c_α is a recursive function. Let C_α denote the extension of c_α to Λ. Then the extension of α to Λ can be characterized (see [7, Theorem 9.5] and [1, Section 5]) as the set

$$\alpha_\Lambda = \{X \mid X \in \Lambda \text{ and } C_\Lambda(X) = 0\}.$$

Combining this with the definition of α_R, we see that

$$(*) \qquad \alpha_R = \{X \mid X \in \Lambda_R \text{ and } C_\alpha(X) = 0\}.$$

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