SUMS OF SMALL NUMBERS OF IDEMPOTENTS

Carl Pearcy and David Topping

1. INTRODUCTION

Recently, Stampfli [16] showed that every (bounded, linear) operator on a separable, infinite-dimensional Hilbert space $\mathscr{H}$ is the sum of 8 idempotent operators. Using this fact, Fillmore [8] gave an elementary proof that every operator on $\mathscr{H}$ is the sum of 64 operators each having square zero, and he also showed that every operator is a linear combination of 257 projections (that is, Hermitian idempotents). These results at first seem somewhat surprising, and since the proofs involve some rather intricate constructions, they do not clearly reveal why the theorems are true.

It is the purpose of this paper to introduce techniques that seem to provide a more natural way of looking at such questions. Using these techniques, which are based partly on the theory of commutators [2], [9], we are able to improve the above-mentioned results considerably, and at the same time to give arguments that are relatively transparent.

In Section 2, we show that most operators on $\mathscr{H}$ (more precisely, every operator in class (F) of [2]) can be written as the sum of four idempotents, and that every operator on $\mathscr{H}$ can be written as the sum of five idempotents. We also show that every operator on $\mathscr{H}$ is the sum of five operators having square zero, and that every Hermitian operator on $\mathscr{H}$ is a real linear combination of eight projections. All of these results remain valid, moreover, on nonseparable spaces.

In Section 3 we take up the question as to which of these constructions can be carried out in the framework of von Neumann algebras, and we show that essentially all of the above results are valid in every properly infinite von Neumann algebra (that is, in every algebra without direct summand of finite type). Our sharpest results are obtained in the case of a factor of type III acting on a separable Hilbert space, where our knowledge of commutators is complete [3]. In such a factor, every nonscalar operator can be expressed as the sum of four idempotents, and also as the sum of four operators each having square zero.

Finally, in Section 4 we consider a certain class of von Neumann algebras, and we show that each algebra of the class can be generated by various small sets of special operators. This is related to the result of Davis [4] that a I_∞-factor on a separable space can be generated by three projections.

2. IDEMPOTENTS, SQUARE-ZERO OPERATORS, AND PROJECTIONS

Throughout Sections 2 and 3, $\mathscr{H}$ will denote a complex, infinite-dimensional, but not necessarily separable, Hilbert space. We denote by $\mathscr{L}(\mathscr{H})$ the algebra of all bounded, linear operators on $\mathscr{H}$. As in [2], we denote by (F) the class of operators obtained by removing from $\mathscr{L}(\mathscr{H})$ all operators of the form $\lambda I + C$, where λ is a

Received October 22, 1966.
The authors acknowledge support from the National Science Foundation. Carl Pearcy is an Alfred P. Sloan Research Fellow.