WILD POINTS OF CELLULAR SUBSETS OF 2-SPHERES IN S^3

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1. INTRODUCTION

In this note we prove that if W is the set of all wild points of a cellular arc that lies on a 2-sphere in S^3, then either W is empty, W is degenerate, or W contains an arc (Theorem 1). Thus there are two types of wild cellular arcs that lie on 2-spheres in S^3: those with exactly one wild point and those with an arc of wild points.

Examples of arcs belonging to each of these two categories have already been described. The arc A constructed on the boundary of a 3-cell C in S^3 by Alford [1] is wild at each of its points. Since the proof for Theorem 5 in [8] also shows that C is cellular, it follows from [12] that A is cellular. Fox and Artin [6] have given an example of a cellular arc on a 2-sphere in S^3 such that the arc has exactly one wild point. It follows from Theorem 2 that a cellular arc on a 2-sphere in S^3 cannot contain two isolated wild points.

In Section 3, we shall show that the results mentioned in the previous paragraphs also hold for cellular finite graphs on 2-spheres in S^3. Section 4 deals with sufficient conditions for certain subsets of 2-spheres to be tame modulo finite sets.

Recently, Burgess [4] gave sufficient conditions for 2-spheres in S^3 to be locally tame except at two points. We make strong use of his results here. In Section 4 of [11], we use the techniques of [4] to prove similar results in a slightly more general setting. Thus, wherever we refer to [4], we could have used [11] instead.

2. DEFINITIONS AND NOTATION

A subset G of S^3 is *cellular* (in S^3) if and only if there exists a sequence $C_1, C_2, \cdots$ of 3-cells in S^3 such that for each positive integer i, $C_{i+1} \subset \text{Int}\, C_i$ and $G = \bigcap_{i=1}^{\infty} C_i$. A *finite graph* is the union of a finite collection of arcs such that if p is a point of intersection of two of these arcs then p is an endpoint of each of the two arcs.

A 2-sphere S in S^3 is *locally tame* at a point p if there exists a disk D on S and a homeomorphism h of S^3 onto itself such that $p \in \text{Int}\, D$ and h(D) is a polyhedron. Furthermore, S is *tame* in S^3 if there exists a homeomorphism h of S^3 onto itself such that h(S) is a polyhedron. We define a subset X of a 2-sphere to be *tame* if X lies on a tame 2-sphere. Also, we say that X is *locally tame* at a point p of X if X lies on a 2-sphere that is locally tame at p. A set X is *locally tame* modulo K if and only if it is locally tame at each point of X - K. We say that a point p of X is a *wild point* of X (or that X is wild at p) if and only if X is not locally tame at p.

A wild point p of K is called an *isolated wild point* of a set K if it lies in an open subset O of K such that K is locally tame at each point of $O - \{p\}$.

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