PSEUDO-ISOTOPIES AND CELLULAR SETS

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In this paper we study some of the relationships between cellular decompositions of manifolds and pseudo-isotopies on manifolds. Our manifolds have no boundary.

A pseudo-isotopy on a manifold N is a homotopy $h: I \times N \to N$ for which each restriction $h_t = h \mid t \times N$ is onto and each h_t is a homeomorphism for $t < 1$. We think of each h_t as a map of N onto N. The map h_1 is said to be the *end* of h, and h is said to end in h_1.

A pseudo-isotopy h *shrinks* the elements of a decomposition D of N if

1. h_0 is the identity,
2. for each point x of N the set $h_1^{-1}(x)$ is an element of D.

Thus h does not collapse subsets of N with reckless abandon. It keeps elements of D separate at all times, and when the shrinking has been carried out at $t = 1$, we find that the pre-image $h_1^{-1}(x)$ of each point is no more and no less than an element of D.

Let f be a map of N onto itself, and let D_f denote $\{f^{-1}(x) \mid x \in N\}$, the decomposition of N induced by f. A pseudo-isotopy on N that shrinks the elements of D_f to points does not necessarily end in f. For example, if f were an orientation-preserving map of a sphere onto itself, there would be no homotopy of f with the identity. One can however prove the following equivalence:

PROPOSITION. *Let* X *be a compact Hausdorff space. A map* f *of* X *onto itself is the end of a pseudo-isotopy on* X *if and only if there exists some pseudo-isotopy on* X *that shrinks the elements of* D_f *to points.*

The proof one way is easy, because if f is the end of a pseudo-isotopy h on X, then $h_0^{-1}h$ is a pseudo-isotopy that shrinks the elements of D_f to points.

The converse follows from the next lemma.

LEMMA. *If* f *and* h_1 *are maps of a compact Hausdorff space onto itself with* $D_f = D_{h_1}$, *then there exists a homeomorphism* g *such that* $gh_1 = f$.

More general forms of this lemma are known, but this one is sufficient here and for the proof of Theorem 2.

A subset A of an n-manifold N is *cellular* if each open subset that contains A also contains a closed n-cell B with $A \subset \operatorname{Int} B$. A cellular set need not be locally connected, nor need it be contractible on itself to a point, or have the fixed-point property. Yet each cellular subset A of N can be shrunk to a point by a pseudo-isotopy on N. That is, there exists a pseudo-isotopy h on N such that h_0 is the identity and A is the only nondegenerate element of D_{h_1}. It is sometimes useful to know that this pseudo-isotopy may be chosen to be the identity outside an arbitrary neighborhood of A. In general, a decomposition of N is called cellular if each of its elements is cellular, and a mapping f defined on N is cellular if D_f is cellular.

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