ON TOPOLOGICAL INFINITE DEFICIENCY

R. D. Anderson

Dedicated to Raymond L. Wilder on his seventieth birthday.

1. INTRODUCTION

Any locally convex, complete, linear metric space is called a Fréchet space. If a Fréchet space is normed, it is a Banach space. The countable infinite product s of lines is an example of a Fréchet space that is not a Banach space. It has recently been shown (see [1], [3], and [5]) that all separable infinite-dimensional Fréchet spaces are homeomorphic to each other. In all of these spaces, as well as in the Hilbert cube I^∞, closed sets of infinite deficiency play an important role. In a Fréchet space X, a closed set K is said to have *infinite deficiency* (or *infinite codimension*) if $X \setminus [K]$ is infinite-dimensional, where [K] denotes the closure of the linear subspace spanned by the elements of K. For the Hilbert cube I^∞, we agree that a closed set K has infinite deficiency if for each of infinitely many different coordinate intervals, K projects onto a single *interior* point of the interval.

As examples of topological theorems dealing with closed sets of infinite deficiency, we mention (i) the result of Klee [7] that in ℓ_2 (the space of square-summable sequences of reals with the norm topology) each homeomorphism between two closed sets of infinite deficiency can be extended to a homeomorphism of ℓ_2 onto itself, and (ii) the result in [2] that if M is a countable union of closed sets of infinite deficiency in s, then $s \setminus M$ is homeomorphic to s. In a sense, a set of infinite deficiency in a space is like a point of the space; but the set itself may be topologically rich, and it may even be homeomorphic to the whole space.

In this paper, we determine what kinds of closed sets have *topological infinite deficiency,* that is, may be carried onto closed sets of infinite deficiency, by some space homeomorphism. Corollary 10.2 gives a characterization of topological infinite deficiency in terms of homotopy properties of the complement of the set. Theorem 10.1 gives similar necessary and sufficient conditions under which a homeomorphism from a closed set onto a closed set of infinite deficiency can be extended to a homeomorphism of the space onto itself. It is worth noting that the same conditions apply to the compact space I^∞ and to each separable, infinite-dimensional Fréchet space.

The methods used in this paper combine refinements of techniques described in [2] with special methods dealing with homotopy properties. Except for Section 9, virtually all the apparatus concerns the Hilbert cube I^∞ and the natural embedding of s as a subset of I^∞. In particular, the apparatus shows (Corollary 10.4) that every homeomorphism of I^∞ onto itself is stable in the sense of Brown and Gluck. This result leads to an affirmative solution of the annulus conjecture for I^∞ (Corollary 10.6).

In Section 11, it is shown that there exists a homeomorphism of I^∞ onto itself carrying the so-called pseudo-boundary into the pseudo-interior.

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