THE BORDISM CLASS OF A BUNDLE SPACE

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Dedicated to R. L. Wilder on his seventieth birthday.

1. INTRODUCTION

In an earlier note [6], it was shown that if M^n is a closed manifold with even Euler characteristic, then the unoriented bordism class $[M^n]_2 \in \mathfrak{N}_n$ can be represented by a manifold fibred differentiably over the circle with structure group Z_2. Now we shall prove that if M^{4m} is a closed oriented manifold with index 0, then modulo an element of order 2 its oriented bordism class is represented by a manifold differentiably fibred over S^2 with structure group SO(2) and an oriented fibre. As a corollary of this, Burdick [2] uses his methods to show that $[M^{4m}]$ is represented, modulo torsion, by a manifold differentiably fibred over S^1 with structure group Z and an oriented fibre that bounds.

The proof of the above result appears in Section 6. We shall briefly outline the steps involved. It was shown by Milnor [9] that $\Omega/p\Omega$, where Ω is the oriented bordism ring of a point, is for any odd prime p a graded polynomial ring over Z_p with a generator in each dimension divisible by 4. A bordism class $[M^{4n}]$ is a generator of $\Omega/p\Omega$ if and only if modulo $p\Omega$ it cannot be expressed as a sum of products of lower-dimensional elements.

In Section 2 we discuss a construction of oriented (or weakly complex) bordism classes. This assigns to a complex (k + 1)-plane bundle $\xi \to M^{2n}$ over a closed oriented manifold the total space $CP(\xi)$ of the associated projective space bundle with fibre CP(k). In (4.1) we find the formula needed to determine whether $[CP(\xi)] \in \Omega_{2(n+k)}$ is a generator of $\Omega/p\Omega$. This is done by computing a numerical invariant in terms of the Chern classes of $\xi \to M^{2n}$.

In Section 4 we obtain a series of corollaries that represent the cases in which an effective computation is possible. In Section 6 a collection of manifolds fibred over S^2 is described. By means of the results established in Section 4 it is shown that some of these manifolds are generators of $\Omega/p\Omega$. The principal result will follow.

Section 5 shows, at least in part, just how effective the construction method of Section 2 is in finding generators of $\Omega/p\Omega$. Previously, several devices have been used in presenting generators, but as far as we know this is the first attempt at determining the efficiency of a construction.

The computations in Section 3 and for (4.1) appear to be tedious. We hope, however, that (4.1) sufficiently unifies this problem to eliminate this kind of work in the future.

Finally, in Sections 2 to 5 we use the weakly complex bordism ring $\mathfrak{U}$ of a point [10], [7, Chapter 1] rather than Ω. It is easier for us to work with, and as we point out in Section 6, there is no loss of generality.

The author expresses his admiration for Professor R. L. Wilder and his distinguished career in topology.

Received May 14, 1966.