SMOOTHNESS CONDITIONS ON CONTINUA IN EUCLIDEAN SPACE

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Dedicated to R. L. Wilder on his seventieth birthday.

One would like an easy way to identify manifolds in the set of topological spaces, and as a step in this direction one may restrict attention to continua in euclidean spaces and use properties of their imbedding as well as intrinsic topological properties. This note is a report on efforts to identify (n - 1)-manifolds in R^n on the basis of both intrinsic and positional properties.

1. FREE CONTINUA

In 1933, Borsuk [5] defined a subset M of R^n to be *free* if for each $\varepsilon > 0$ there exists a continuous map $f: M \to R^n$ such that each point is moved a distance less than ε and $f(M) \cap M = \emptyset$. He asked whether a free locally contractible continuum M that separates R^n must be an (n - 1)-manifold. It is clear that local contractibility (or some form of local connectedness) is needed, because the Warsaw Circle (Figure 1) is free in R^2 but is surely not a manifold.

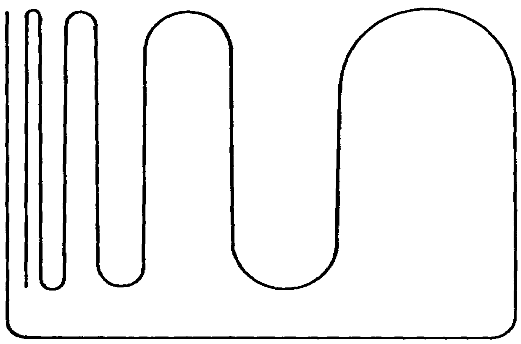

Figure 1

In the same year, Wilder [19] showed that such an M in R^2 or R^3 is a manifold (of dimension one or two, respectively). For the higher dimensions he proved that M is an (n - 1)-dimensional generalized closed (homology) manifold. This means that homologically it has the local properties of a manifold. Here is Wilder's original definition (in which the geometrical meaning of the definition is more apparent than in more recent definitions based on cohomology and sheaf theory).

We use Čech homology with coefficients in a field F. The space X (assumed to be separable and metric, for convenience) is an n-dimensional generalized manifold if it has covering dimension n and satisfies the following four conditions.

(i) $H_n(X; F) \cong F$, but if Y is any proper closed subset of X, then $H_n(Y; F) = 0$.

(ii) All sufficiently small cycles bound.

(iii) For each $x \in X$ and each $\varepsilon > 0$, there exist δ and η such that $0 < \eta < \delta < \varepsilon$ and each i-cycle $(1 \leq i \leq n - 2)$ on the "sphere"

$$S(x, \delta) = \{y \mid \rho(x, y) = \delta\}$$

bounds in the "annulus" $B(x, \varepsilon) - B(x, \eta)$.

(iv) An (n - 1)-cycle on $S(x, \delta)$ bounds in $X - B(x, \eta)$.

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