THE HOMOTOPY EXCISION THEOREM

E. Spanier

Dedicated to Professor R. L. Wilder on his seventieth birthday.

1. INTRODUCTION

This paper is devoted to a study of the homomorphism of homotopy groups induced by an inclusion map

$$e: (B, C) \subset (X, A)$$

such that $B - C = X - A$. Such an inclusion map is called an *excision map*, and we see that $C = A \cap B$ and $X = A \cup B$, so that an excision map is an inclusion map of the form $(B, A \cap B) \subset (A \cup B, A)$. In case A and B are open sets, it is well known [4, pp. 199-200], [6, p. 189] that the excision map induces isomorphisms of all the corresponding singular homology and cohomology groups; however, it need not induce isomorphisms of the corresponding homotopy groups.

We present an example to illustrate this. For a space Y, let SY be the join of Y with a pair of points p, p', and let $S: \pi_q(Y) \to \pi_{q+1}(SY)$ be the suspension homomorphism (thus, if $\alpha: S^q \to Y$ represents $[\alpha] \in \pi_q(Y)$, then $S[\alpha]$ is represented by the composite of a fixed homeomorphism $S^{q+1} \approx SS^q$ with $S\alpha: SS^q \to SY$). Then $SY - p$ and $SY - p'$ are contractible, $SY - (p \cup p')$ has the same homotopy type as Y, and there is a commutative diagram

$$\begin{array}{ccc} \pi_{q+1}(SY - p', (SY - p) \cap (SY - p')) & \xrightarrow{e_\#} & \pi_{q+1}(SY, SY - p) \\ \approx \downarrow & & \uparrow \approx \\ \pi_q(SY - (p \cup p')) \approx \pi_q(Y) & \xrightarrow{S} & \pi_{q+1}(SY) . \end{array}$$

It follows that $e_\#: \pi_{q+1}(SY - p', (SY - p) \cap (SY - p')) \to \pi_{q+1}(SY, SY - p)$ is an isomorphism if and only if $S: \pi_q(Y) \to \pi_{q+1}(SY)$ is also an isomorphism. Since the suspension homomorphism is not generally an isomorphism, neither is the homomorphism induced on homotopy groups by an excision map.

On the other hand, with suitable hypotheses an excision map does induce an isomorphism in homotopy for a certain range of dimensions, and this result can be used to prove that the suspension homomorphism is an isomorphism for a corresponding range of dimensions.

A pair (X, A) is said to be n-*connected* for $n \geq 0$ if for $q \leq n$ every map $(E^q, S^{q-1}) \to (X, A)$ is homotopic relative to S^{q-1} to a map sending all of E^q into A. The following is the general result proved in this note.

HOMOTOPY EXCISION THEOREM. *Let* A, B *be subsets of a space* $X = A \cup B$ *such that* $(A, A \cap B)$ *is* n-*connected and* $(B, A \cap B)$ *is* m-*connected, where* $n, m \geq 0$. *If either*

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