VECTOR FIELDS AND CHARACTERISTIC NUMBERS

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Dedicated to R. L. Wilder on his seventieth birthday.

1. INTRODUCTION

A well-known theorem of Heinz Hopf asserts that on a compact manifold the properly counted number of zeros of a vector field equals the Euler number of the manifold. The purpose of this paper is to show that when a vector field satisfies certain differential equations, then there are other relations between the characteristic numbers of the manifold and local invariants of the vector field near its zeros.

The two cases of greatest interest are (i) *where the vector field is holomorphic and* (ii) *where it defines an infinitesimal motion of a Riemannian manifold.* In the first case all the characteristic numbers will be seen to be determined by the zeros of the vector field. In the second case the Pontrajagin numbers are determined—and of course the Euler numbers—but the Stiefel-Whitney numbers are not. In short, the local behaviour at the zeros of the vector field determines all the rational characteristic numbers in both cases.

To describe these relations explicitly, when the vector field X behaves generically at its zeros, we recall first that the Lie derivative in the direction of X is a well-defined differential operator $\mathfrak{L}(X)$ on all the tensor fields over M, and that it *has order* 0 *at the zeros of* X.

Thus, in particular, $\mathfrak{L}(X)$ induces a linear map L(X) of the tangent space TM to M restricted to the set zero(X):

$$L_p(X) = \mathfrak{L}(X) \,|\, T_p M \qquad (p \in \text{zero}(X)).$$

The vector field X will be called *nondegenerate* if L(X) is nonsingular at all the zeros of X, and the eigenvalues of L(X) will be referred to as the *characteristic roots* of X at its zeros.

Second, we recall that a complex structure on M endows the real tangent bundle $T_R M$ of M with a complex structure. The Chern classes of this C-bundle are therefore well-determined elements $c_i(M) \in H^{2i}(M, Z)$. Suppose now that $\Phi(c) = \Phi(c_i, \cdots, c_m)$ $(m = \dim_C M)$ is a polynomial in the indeterminates c_i with complex coefficients. By replacing c_i with $c_i(M)$, we obtain a cohomology class $\Phi\{c(M)\}$ in $H^*(M; C)$ whose value on the orientation class [M] will be denoted by $\Phi(M)$. Of course, we define the value of a class $u \in H^*(M; C)$ on [M] to be zero if u contains no elements of degree equal to the dimension of M. Hence $\Phi(M) = 0$ unless Φ involves monomials of "*weight* m" in the c_i, that is, expressions of the form

$$w_a = c_1^{a_1} c_2^{a_2} \cdots c_n^{a_n} \qquad (a_1 + 2a_2 + \cdots + n a_n = m).$$

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