CONCORDANCE OF DIFFERENTIABLE STRUCTURES—TWO APPROACHES

James R. Munkres

Dedicated to Professor R. L. Wilder on his seventieth birthday.

The following two basic problems in differential topology have attracted considerable attention in the last few years.

I. Given a piecewise-linear manifold K, find for it a compatible differentiable structure α.

II. Classify such structures, up to diffeomorphism or some other suitable equivalence relation.

A *piecewise-linear manifold* is a complex K that is locally piecewise-linearly homeomorphic to euclidean space R^n. *Compatibility* means that for some subdivision of K, each simplex σ of the subdivision inherits its usual differentiable structure; we restrict ourselves to structures that are compatible.

There are several possible equivalence relations one might study; the one of particular interest to us is that of concordance: Two differentiable structures α and β on K are said to be *concordant* if there exists a differentiable structure γ on $K \times I$ that equals α on $K \times 0$ and β on $K \times 1$. (The structure γ is called a *concordance* between α and β; it is a *strong concordance* if each level manifold $K \times t$ is a differentiable submanifold of $(K \times I)_\gamma$.) Concordance is a natural equivalence relation, in the sense that it establishes a connection between our two problems—constructing a concordance is simply the problem of finding a differentiable structure on $K \times I$ that extends a preassigned differentiable structure on the boundary. The question of the relation of concordance to diffeomorphism we leave in abeyance for the moment.

Our purpose is to give a brief survey of the subject, to outline the two main attacks that have been made on these problems, to indicate the connections between them, and to suggest some promising directions for future investigation.

1. THE GEOMETRIC APPROACH

The first of these attacks is the geometric one we made a few years ago on Problem II [17]. It involves the groups Γ_n, defined earlier by Thom in his own attacks (not completely successful) on these problems. Γ_n is defined as the group of diffeomorphisms of S^{n-1}, modulo the subgroup consisting of those extendable to diffeomorphisms of the ball B^n; it is abelian, and it has been proved to vanish for $n \leq 4$ [2], [16], [22].

Our general approach is the following: We attempt to classify differentiable structures on K — up to diffeomorphism for the moment. Our way of proceeding is to take a fixed structure α on K, and any other structure β, and to try to construct a diffeomorphism of K_α with K_β. The obstructions that occur provide some

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