NEIGHBORHOODS OF SURFACES IN 3-MANIFOLDS

D. R. McMillan, Jr.

Dedicated to R. L. Wilder on his seventieth birthday.

1. INTRODUCTION

Let S be a closed (that is, compact and boundaryless) 2-manifold topologically embedded in a two-sided manner in Int M, where M is a piecewise linear 3-manifold. The main result in this paper (Theorem 2) is that, arbitrarily close to S, there exists a polyhedral neighborhood of S, homeomorphic to $S \times [0, 1]$ with finitely many "small" handles of index 1 attached. In particular, if S is orientable, some neighborhood of S is embeddable in Euclidean 3-dimensional space E^3. In this sense, we can study many pathological embeddings in 3-manifolds without leaving E^3.

These results continue the line of investigation begun in [15] (see [16] for a survey of the results to be found in both papers), and we rely on some of that work, as well as on many of R. H. Bing's theorems (references [2] to [9]). We are also indebted to Professor Bing for many helpful discussions on these topics.

Using the above notation, and assuming that M - S has components U_0 and U_1, we say that S is *locally tame from* U_0 *at* $p \in S$ if the closure of U_0 is a topological 3-manifold at p. If the closure of U_0 is a 3-manifold, we say that S is *tame from* U_0. The term "manifold" will always refer to a *connected* set. When we wish to emphasize that a manifold possesses a combinatorial triangulation, we shall use the prefix "piecewise linear" (abbreviated: pwl), even though each topological manifold of dimension 3 or less is known to be a piecewise linear manifold. By a *cube-with-handles*, we mean a 3-manifold homeomorphic to the regular neighborhood in E^3 of a finite, connected graph. In considering a mapping $f\colon X \times [0, 1] \to Y$, we shall sometimes use the notation $f_t\colon X \to Y$ $(t \in [0, 1])$ to mean the mapping defined by $f_t(x) = f(x, t)$. Similar notation will refer to an f with domain $X \times [-1, 1]$.

By a *null-sequence* $E_1, E_2, \cdots$ of subsets of a metric space we mean a sequence such that the diameters of its elements converge to zero. Let S be a closed 2-manifold topologically embedded in Int M, where M is a piecewise linear 3-manifold. Let $X \subset S$ be a closed set, and let $U_1, U_2, \cdots$ be the components of S - X. We shall call X an *S-curve* if $\overline{U}_1, \overline{U}_2, \cdots$ is a null-sequence of mutually exclusive 2-cells with $\bigcup_i U_i$ dense in S. In case S is a 2-sphere, such an X is called a *Sierpinski curve* (see [5, Section 3]). We call

$$S - \bigcup_i \overline{U}_i \subset X$$

the *inaccessible part* of X. We shall say that an S-curve X is *tame* in M if for each 2-manifold J that is homeomorphic to S, contains X, and is locally tame at each point of J - X, it follows that J is tame in M. If S is a 2-sphere, then a

Received February 16, 1966.

This research was supported in part by grant NSF GP-4125. The author is an Alfred P. Sloan Fellow.