IMPROVING THE INTERSECTIONS OF LINES AND SURFACES

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Dedicated to R. L. Wilder on his seventieth birthday.

Suppose S^2 is a topological 2-sphere and L^1 is a straight line in Euclidean 3-space E^3. If one had an easy proof of the following corollary, one could use the corollary to simplify the proof of the Approximation Theorem (Theorem 7 of [2]) and the Side Approximation Theorem (Theorem 16 of [6]).

COROLLARY TO THEOREM 2. *For each* $\varepsilon > 0$ *there exists a homeomorphism* $h: E^3 \to E^3$ *such that* $L^1 \cap h(S^2)$ *is finite and* $h = I$ *(identity) outside an* ε*-neighborhood of* $S^2 \cap L^1$.

We do not get a proof of the corollary that makes it useful in proving these theorems, since we use consequences of the Side Approximation Theorem to the hilt in proving Theorem 2, the basis for the corollary. There seems to be good reason for giving a proof of Theorem 2 even if it is complicated, since aside from its intrinsic interest, Theorem 2 is needed in proving some other interesting theorems, such as the result by McMillan (Theorem 2 of [8]) that each 2-sphere topologically embedded in an arbitrary 3-manifold has a neighborhood that can be embedded in E^3.

We denote the distance function by ρ. If f_1, f_2 are two maps of the space X into the metric space Y, we use $\rho(f_1, f_2)$ to denote the least upper bound $\rho(f_1(x), f_2(x))$ $(x \in X)$.

THEOREM 1. *Suppose* S_1, S_2 *are two 2-spheres in* E^3, S_2 *is tame,* $\varepsilon > 0$, *and* X *is a tame Sierpinski curve in* S_1 *such that each component of* $S_1 - X$ *has diameter less than* ε. *Then there exists an isotopy* h_t $(0 \leq t \leq 1)$ *of* E^3 *onto itself such that* $h_0 = I$, *the set* $S_2 \cap h_1(X)$ *is the union of a finite number of mutually exclusive simple closed curves each in the inaccessible part of* $h_1(X)$, $h_1(X)$ *locally lies on different sides of* S_2 *near these simple closed curves,* $h_t = I$ *outside the* ε*-neighborhood of* $S_1 \cap S_2$, *and* $\rho(h_t, I) < \varepsilon$.

Proof. Let δ be a positive number such that each component of $S_1 - X$ has diameter less than $\varepsilon - 4\delta$.

Let S_1' be a tame 2-sphere obtained by replacing each component of $S_1 - X$ by the interior of a tame disk homeomorphically so close to the component that the tame disk has diameter less than $\varepsilon - 4\delta$ and each point of $S_1' \cap S_2$ lies within δ of $S_1 \cap S_2$. The Approximation Theorem (Theorem 7 of [2]) tells us that this replacement is possible, and Theorem 8.2 of [7] assures us that S_1' is tame.

Since S_2 is tame, we suppose it is polyhedral.

Since S_1' is tame, there exists a homeomorphism $g: K^2 \times [-1, 1] \to E^3$, where K^2 is a polyhedral 2-sphere and $g(K^2 \times 0) = S_1'$. Using Theorem 9 of [3] or Theorem 2 of [9], we learn that g can be chosen to be locally piecewise linear on $K^2 \times (0, 1]$. Let f_t $(0 \leq t \leq 1)$ be an isotopy of $K^2 \times [-1, 1]$ onto itself that is fixed outside a small neighborhood of $g^{-1}(S_1' \cap S_2)$ and pushes a small neighborhood of $g^{-1}(S_1' \cap S_2)$ in $K^2 \times 0$ onto a polyhedron on the positive side of $K^2 \times 0$. By considering $g f_t g^{-1}$

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