OPEN MAPS ON HAUSDORFF SPACES

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Dedicated to R. L. Wilder on his seventieth birthday.

In [1], McAuley proved the following theorem.

THEOREM 1. *Suppose that* X *is a compact subset of a metric space* M, Bd X $\neq \emptyset$, Int X $\neq \emptyset$, *and* f *is a light open mapping of* X *into* M *such that*

(1) f(Int X) = Int f(X),

(2) f(Bd X) = Bd f(X),

(3) *the singular set* S_f *has the property that* $f(S_f)$ *does not contain a nonempty set open relative to* f(X),

(4) $f(S_f)$ *does not separate* f(X), *and*

(5) *there exists a nonempty* U *in* X, *open relative to* X, *such that* $f \mid U$ *is one-to-one and* $f^{-1} f(U) = U$.

Then f *is a homeomorphism.*

In this note we show that McAuley's methods yield the same conclusions with weaker hypotheses. Given topological spaces X and Y and a map f: X → Y, we define the sets

$$S_f = \{x \mid f \text{ is not one-to-one on any neighborhood of } x\},$$

$$P = \{y \mid f^{-1}(y) \text{ is nondegenerate}\}.$$

We recall that f is *open* if and only if for each set U open in X the set f(U) is open in f(X). The following three lemmas are known and obvious.

LEMMA 1. S_f *is closed.*

LEMMA 2. *If* X *is a Hausdorff space and* f *is an open map, then* P *is an open subset of* f(X).

LEMMA 3. *If* X *is a compact Hausdorff space,* Y *is a Hausdorff space, and* f *is an open map, then* $P \cup f(S_f)$ *is closed.*

THEOREM A. *Let* f: X → Y *be an open map of a compact Hausdorff space* X *into a Hausdorff space* Y *such that*

(1) $f(S_f) \not\supset P$ *(unless each is empty),*

(2) $f(S_f)$ *does not separate* f(X), *and*

(3) $P \cup f(S_f) \neq f(X)$.

Then f *is a homeomorphism.*

It is clear that this is a generalization of Theorem 1, since hypotheses (3), (4), and (5) of Theorem 1 imply hypotheses (1), (2), and (3), respectively, of Theorem A.

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