ε-TAMING IN CODIMENSION 3

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Let h: M → Int N be a homeomorphism of a closed combinatorial k-manifold into the interior of a combinatorial n-manifold. For $n \geq 2k + 2$, Cantrell and Edwards [2] have shown that h is locally flat if h is almost locally flat, and Gluck [5] has shown that h is ε-tame if it is locally flat. More recently, Černavskiĭ [3] announced that if $h: M \to E^n$ is almost locally flat, then h is ε-tame if $n > \frac{3}{2}k + 1$, and h is locally flat if $n - k \neq 2$ and $n \geq 5$. Our main result is that h: M → Int N is ε-tame if h is almost locally piecewise linear and $n - k \geq 3$. Furthermore we avoid using Černavskiĭ's result in this codimension. The author would like to thank T. Homma for several enlightening discussions.

Before proceeding with the proof, we give the following definitions. By a *manifold* we mean a combinatorial manifold with boundary. Let Int N and Bd N denote the interior and boundary, respectively, of the manifold N. If A is a closed subset of N and ε is a positive number, then by an *ε-push* of (N, A) we mean an isotopy $H: N \times [0, 1] \to N$ such that H_0 is the identity mapping 1, for $t \in [0, 1]$ the restriction $H_t \mid N - U_\varepsilon(A)$ is the identity, and $\operatorname{diam}[H(x \times [0, 1])] < \varepsilon$ for all $x \in N$. An embedding h: M → Int N is said to be *ε-tame* if for each $\varepsilon > 0$ there exists an ε-push H^ε of (N, h(M)) such that $H_1^\varepsilon h: M \to N$ is piecewise linear (pwl). The embedding h is said to be locally piecewise linear at $p \in M$ if there exists a neighborhood U of p such that $h \mid U$ is pwl.

Denote by $B^n(r)$ the set

$$\{(x_1, x_2, \cdots, x_n) \in E^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 \leq r^2\}.$$

Let A be an n-cell, B a k-cell, with Bd B ⊂ Bd A and Int B ⊂ Int A; then (A, B) is a *cell pair of type* (n, k). The cell pair (A, B) of type (n, k) is said to be *trivial* if (A, B) is homeomorphic to $(B^n(1), B^k(1))$. (Throughout the discussion, we consider E^m as being embedded in the natural way in E^n). A homeomorphism h: M → Int N is said to be *locally flat* at $p \in \text{Int } M$ if there is a neighborhood U of p in M and a neighborhood V of h(p) in N such that $(V, V \cap h(M)) = (V, h(U))$ is a trivial cell pair of type (n, k).

For reference we state Zeeman's engulfing theorem.

THEOREM (Zeeman [11]). *Let* M *be a* k-*connected* m-*manifold* ($k \leq m - 3$), *and let* X *and* K *be subpolyhedra of* Int M *such that* X *is collapsible and* K *is* k-*dimensional. Then there exists a* (k + 1)-*dimensional subpolyhedron* L *with* K ⊂ L ⊂ Int M *such that* X ∪ L *is collapsible.*

Denote by S(f) the singular set $\text{Cl}\{x \in M \mid f^{-1}f(x) \neq x\}$ of the map f: M → N, and by C(P) the abstract cone over the polyhedron P.

LEMMA 1. *Let* $h: \Delta \to E^n$ *be a homeomorphism of a* k-*simplex into* E^n *such that* h *is locally* pwl *except at* $p \in \text{Int } \Delta$. *If* $n - k \geq 3$, *then there exists a*

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