ON THE BOUNDARY BEHAVIOR OF CONFORMAL MAPS

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1. INTRODUCTION AND SUMMARY

Let f be schlicht in $|z| < 1$, and suppose that $c = \lim_{x \to 1-0} f(x) \neq \infty$ exists. Then $f(z) \to c$ as $z \to 1$ in any Stolz angle, hence also as $z \to 1$ in some domain G that is tangential at $z = 1$ (by this we mean that G contains every Stolz angle in a sufficiently small neighborhood of $z = 1$). In other words,

$$f(z) \to c \quad \text{as } z \to 1 \ (z \in G), \tag{1}$$

and as a consequence

$$\text{the image } f(G) \text{ has finite area.} \tag{2}$$

Note, however, that G depends on f. George Piranian raised the following question: Does there exist a domain G tangential at $z = 1$ that is *independent* of f and for which (1) or (2) holds. For example: Does (1) or (2) always hold for $G = \{|z - 1/2| < 1/2\}$?

Our answer is negative in both cases.

THEOREM 1. *Let* $\{z_p\}$ *be any sequence with* $|z_p| < 1$ *such that* $z_p \to 1$ *and* $\arg(z_p - 1) \to \pi/2$ $(p \to \infty)$. *Then there exists a function* f *that is bounded and schlicht in* $|z| < 1$, *for which* $\lim_{x \to 1-0} f(x)$ *exists but* $\{f(z_p)\}$ *diverges.*

THEOREM 2. *Let* G *be any subdomain of* $|z| < 1$ *that is tangential at* $z = 1$. *Then there exists a function* f, *schlicht in* $|z| < 1$, *for which* $\lim_{x \to 1-0} f(x) \neq \infty$ *exists but* $f(G)$ *has infinite area.*

2. PRELIMINARY RESULTS

It will be more convenient to work in the half-plane $\Re z > 0$ than in the disk $|z| < 1$. So we assume that $\{z_p\}$ is a sequence with $\Re z_p > 0$ and $z_p \to 0$, $\arg z_p \to \pi/2$ $(p \to \infty)$.

Let $0 < h < 1$, and let D be a simply connected domain in the half-plane $u > h$ of the $w = u + iv$-plane that is symmetric with respect to the real axis, contains the point $w = 1$ and some rectangle $\{h < u < h + a,\ |v| < 1\}$. For $0 < \varepsilon < 1$, let D_ε consist of the rectangle $R = \{0 < u < h,\ -4 < v < 4\}$ and of D, the two domains being connected by an opening of width 2ε; that is, let

$$D_\varepsilon = R \cup (h - i\varepsilon,\ h + i\varepsilon) \cup D.$$

Let the schlicht function $w = f_\varepsilon(z)$, normalized by the conditions $f_\varepsilon(0) = 0$ and $f_\varepsilon(1) = 1$, map the half-plane $\Re z > 0$ onto D_ε.

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