ON THE COEFFICIENTS OF UNIVALENT FUNCTIONS

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1. STATEMENT OF RESULTS

We shall show that the trivial estimates $a_n = o(1/\sqrt{n})$ and $b_n = o(1/\sqrt{n})$ for the coefficients of bounded univalent functions and meromorphic univalent functions, respectively, are not essentially best possible.

THEOREM 1. *Let*

$$g(z) = z + b_0 + \cdots + b_n z^{-n} + \cdots$$

be analytic and univalent in $1 < |z| < \infty$. *Then*

$$\text{(1.1)} \qquad \int_0^{2\pi} |g'(\rho e^{i\theta})|\, d\theta \leq A\left(1 - \frac{1}{\rho}\right)^{-\frac{1}{2}+\frac{1}{300}} \qquad (1 < \rho < \infty),$$

$$\text{(1.2)} \qquad |b_n| \leq A n^{-\frac{1}{2}-\frac{1}{300}},$$

where A *is an absolute constant.*

The only previously known estimate, $|b_n| \leq n^{-1/2}$, follows immediately from the area theorem. In the opposite direction, the first nontrivial result was due to Clunie [1], who constructed a univalent function for which $|b_n| > n^{0.02-1}$ for infinitely many n. This was recently improved [8] to $|b_n| > n^{0.139-1}$.

Let γ be the smallest number such that

$$|b_n| \leq A(\varepsilon) n^{\gamma+\varepsilon-1}$$

for every $\varepsilon > 0$. The estimates above imply that

$$\text{(1.3)} \qquad 0.139 \leq \gamma < 0.497.$$

The true value of γ is unknown.

Remark. We can prove an estimate that is slightly stronger than (1.2): For $5 < \lambda < \infty$,

$$\text{(1.4)} \qquad \sum_{k=1}^{n} k^{\lambda} |b_k|^{\lambda} \leq A(\lambda) n^{\frac{\lambda}{2}-\frac{1}{2}+\frac{18}{\lambda-1}} \qquad (n = 1, 2, \cdots),$$

where $A(\lambda)$ is independent of g.

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