DIFFERENTIABLE TRANSFORMATION GROUPS ON HOMOTOPY SPHERES

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1. INTRODUCTION

The purpose of the present paper is to use techniques and results of differentiable topology in the study of differentiable transformation groups. Although our objective is the same as that in recent works of Conner and Floyd (see, for example, [4]), we do not use bordism theory, a fundamental tool employed by Conner and Floyd, but rely on pasting techniques instead. Recent papers by W. C. Hsiang and W. Y. Hsiang on differentiable actions have some points of contact with our work.

Denote by S^n the unit n-sphere in euclidean $(n+1)$-space, and regard S^{n-1} as a subspace of S^n by identifying every $(x_1, \cdots, x_n) \in S^{n-1}$ with $(x_1, \cdots, x_n, 0) \in S^n$. Let G be a compact Lie group acting as a transformation group on S^n, and let

$$F = \{x \in S^n \mid Gx = x\};$$

that is, let F be the set of the fixed points of G in S^n. If the action is linear, it is obvious that F is diffeomorphic to S^r for some integer r $(-1 \leq r \leq n)$. (It is understood that S^{-1} denotes the null set.) Moreover, there is a diffeomorphism of S^n onto itself that maps F onto S^r.

Suppose that the action of G on S^n is only differentiable. Then F is a (differentiable) submanifold of S^n. However, F may not be homeomorphic to a sphere, because F may not even be an integral cohomology sphere [2]. In this paper we shall study differentiable actions of G on S^n in which F is an integral cohomology sphere. They are more general than linear actions, but the case just mentioned is excluded.

Let G be a nontrivial compact Lie group and n(G) the integer such that G is isomorphic to a subgroup of the orthogonal group O(n(G)) but not to any subgroup of the orthogonal group O(n(G) - 1). It is easily seen that if G acts effectively and differentiably on S^n, the fixed point set F is of dimension at most n - n(G) (Proposition 1). As a modification of a theorem of Montgomery and Samelson [7], we shall show that for any nontrivial compact Lie group G, there are infinitely many effectively differentiable actions of G on $S^{n(G)+3}$ of which the fixed point sets are integral cohomology 3-spheres with fundamental groups not isomorphic to one another (Theorem 1). Hence the fixed point set of an effective differentiable action of any nontrivial compact Lie group on S^n may not be homeomorphic to a sphere, even when it is an integral cohomology sphere attaining the highest possible dimension.

We next consider the case when G is the circle group SO(2). As we said in the preceding paragraph, it is possible to have an effective differentiable action of G on S^5 such that the fixed point set F is an integral cohomology 3-sphere that is not simply connected. (Notice that n(SO(2)) = 2, so that n(SO(2)) + 3 = 5.) Therefore such an action is not equivalent to a linear action. In this particular case, it is

Received February 12, 1966.
The second author was supported in part by the U. S. Army Research Office.