INTEGRABILITY CONDITIONS FOR ALMOST-COMPLEX MANIFOLDS

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The integrability condition $\bar{\partial}^2 = 0$ characterizes those almost complex structures that arise from complex-analytic structures, and it should in some way be reflected in corresponding conditions on Riemannian metrics; that is, we should be able to distinguish between metrics that are almost-hermitian with respect to integrable almost-complex structures, and those that are almost-hermitian with respect to non-integrable structures. This paper introduces a real-valued function (on the tangent vectors of the manifold in question) whose positivity (or lack of it) permits us to make the distinction.

More precisely, let J be an almost-complex structure on the 2n-dimensional manifold M, and let g(X, Y) be an almost-hermitian metric on M such that

$$g(JX, JY) = g(X, Y).$$

We denote by B the bundle of almost-complex frames of M, by ω the restriction to B of the Riemannian connection of M, and by $\mathfrak{M}$ the orthogonal complement to the Lie algebra of U(n) in the Lie algebra of O(2n). Let $\triangle$ denote the $\mathfrak{M}$-component of ω, and finally, for each tangent vector X of M, let

$$\sigma_g(X) = -\operatorname{trace} \operatorname{Im} [\triangle(X), \triangle(JX)]$$

(see Section 3). We shall prove in Section 5 that σ_g is nonnegative whenever J is integrable, and in Section 6 that σ_g vanishes identically if and only if g is a Kähler metric.

1. THE METRICS

Let M be an almost-complex manifold of real dimension 2n, with an almost-complex operator J: $J^2 = -I$. Let g(X, Y) be a Riemannian metric on M that is compatible with J. That is, let

$$g(JX, JY) = g(X, Y)$$

for each pair of tangent vectors X and Y of M; or equivalently, let

$$g(JX, Y) = -g(X, JY)$$

for each pair of tangent vectors X and Y of M. Such a metric g is called *almost-hermitian*, and its *fundamental form* is the real-valued 2-form

$$\Omega(X, Y) = -(4\pi)^{-1} \cdot g(X, JY).$$

Received November 22, 1965.