PRIMES IN ARITHMETIC PROGRESSIONS

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For any positive integer q and any integer a relatively prime to q, let

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \Lambda(n),$$

where $\Lambda(n)$ is the arithmetical function which is log p if n is a power of a prime p and 0 otherwise. The prime number theorem for arithmetic progressions, in the form which Walfisz deduced from Siegel's result on L-functions, states that if $q \leq (\log x)^N$ for each fixed N, then

$$\psi(x; q, a) = \frac{x}{\phi(q)} + O(x e^{-C\sqrt{\log x}}) \tag{1}$$

for some positive constant C depending on N.

In an important recent paper [1], Bombieri has investigated the behaviour of the error term in this theorem as q varies, up to $x^{1/2}(\log x)^{-B}$ for some fixed positive B. He defined

$$E(x, q) = \max_a |\psi(x; q, a) - x/\phi(q)|, \qquad E^*(x, q) = \max_{y \leq x} E(y, q),$$

and he proved in his Theorem 4 that for any fixed positive A there exists a positive constant B such that, if $X \leq x^{1/2}(\log x)^{-B}$, then

$$\sum_{q \leq X} E^*(x, q) \ll \frac{x}{(\log x)^A}.$$

(We use Vinogradov's notation $\ll$ to indicate an inequality with an unspecified constant factor.) Bombieri's proof was based on a general theorem of the 'large sieve' type (his Theorem 3), but it employed also a whole range of methods and techniques from analytic number theory. In particular, the crucial step was the proof of a density theorem on the zeros of L-functions.

The object of the present paper is to establish another result on the average of the error term

$$\psi(x; q, a) - x/\phi(q);$$

this result can be deduced very simply from a general large sieve theorem similar to Bombieri's Theorem 3, which we have proved elsewhere [2, Corollary 2 to Theorem 4].

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