QUASI-ISOMORPHISM OF PRIMARY GROUPS

Paul Hill

The primary groups G and H are said to be quasi-isomorphic if there exist a subgroup G^* of G and a subgroup H^* of H such that G/G^* and H/H^* are bounded and $G^* \cong H^*$. Beaumont and Pierce have recently given necessary and sufficient conditions on the Ulm invariants in order that G and H be quasi-isomorphic in case G and H are countable [1] and in case G and H are direct sums of cyclic groups [2]. The conditions of Beaumont and Pierce are (equivalent to) the following:

(i) there exists an integer $k \geq 0$ such that for all integers $n \geq 0$ and $r \geq 0$

$$\sum_{j=n+k}^{n+k+r} \mathscr{F}_G(j) \leq \sum_{j=n}^{n+2k+r} \mathscr{F}_H(j), \qquad \sum_{j=n+k}^{n+k+r} \mathscr{F}_H(j) \leq \sum_{j=n}^{n+2k+r} \mathscr{F}_G(j),$$

where $\mathscr{F}_G$ is the Ulm function defined on G.

(ii) $p^\omega G \cong p^\omega H$.

It was shown in [1] that conditions (i) and (ii) are always necessary in order that G and H be quasi-isomorphic for arbitrary primary groups G and H. The proof of this is indeed simple. However, the proof in [1] of the sufficiency of (i) and (ii) for the countable case is rather laborious, and the proof in [2] of the sufficiency of (i) for direct sums of cyclic groups is based on the countable case. Much simpler proofs are given below; in fact, the results of [1] and [2] are immediate consequences of Lemma 1. Moreover, we are able to extend these results to direct sums of countable groups and beyond.

We write $G \doteq H$ to mean that G and H are quasi-isomorphic.

THEOREM. *Suppose that* G *and* H *are primary groups such that* $G/p^\omega G$ *and* $H/p^\omega H$ *are direct sums of cyclic groups. Then conditions* (i) *and* (ii) *are necessary and sufficient in order that* $G \doteq H$.

As we have mentioned, the necessity of (i) and (ii) is readily established in [1]. Thus we are concerned only with the sufficiency.

LEMMA 1. *Suppose that* G *and* H *are direct sums of cyclic groups and that condition* (i) *holds. If* $\mathscr{F}_G(0) = |G|\,|H|\,\aleph_0 = \mathscr{F}_H(0)$, *then there exists an isomorphism from* $G[p]$ *onto* $H[p]$ *that alters heights (computed in* G *and* H) *no more than* k.

Proof. For each nonnegative integer n, let A_n be a closed initial segment of ordinal numbers such that the segment has cardinality $\mathscr{F}_G(n)$. Similarly, let B_n be a closed initial segment of ordinals such that the segment has cardinality $\mathscr{F}_H(n)$. Define

$$A = [(n, x)\colon 0 \leq n < \omega \text{ and } x \in A_n], \qquad B = [(n, x)\colon 0 \leq n < \omega \text{ and } x \in B_n],$$

and consider the lexicographical order on A and B. Call the first component n of the element (n, x) in A or B the *index* of the element. For each nonnegative

Received May 28, 1966.