THE JACOBSON RADICAL OF A GROUP ALGEBRA

Michael I. Rosen

Introduction. Let K be a field of characteristic zero, and G a group. It has been conjectured that the group algebra K[G] is semisimple, in other words, that the Jacobson radical of K[G] is zero. If K has elements that are transcendental over the field Q of rational numbers, then K[G] is indeed semisimple (see Amitsur [1]). For the case where K is algebraic over Q, only partial results are known. For example, if G is abelian or G/C is locally finite (C being the center of G), then the conjecture is true (see [1], [4], [5]). We shall give new proofs of these results, and we shall verify the conjecture for a much larger class of groups.

By a *linear representation* p of a group G we shall mean a homomorphism from G to a finite-dimensional linear group over some field. G is said to be *residually linear* if for every $g \in G$ $(g \neq e)$ there exists a linear representation p such that p(g) is not the identity. Our main result is the following.

THEOREM A. *If every finitely generated subgroup of* G *is residually linear, then* K[G] *is semisimple.*

In particular, the result applies to any linear group. Clearly, the property of being residually linear is inherited by subgroups. By the Peter-Weyl theorem, a compact group is residually linear. This gives the following result.

COROLLARY. *Let* H *be a subgroup of a compact group. Then* K[H] *is semisimple.*

The limitations of our methods will be shown in Section 3, where we prove the following proposition.

THEOREM B. *Let* G *be an infinite, finitely generated, simple group. Then* G *has no nontrivial linear representations over any field.*

Graham Higman [2] has shown that there exist groups satisfying the hypotheses of Theorem B.

1. LEMMA 1.1. *Let* S *be a ring, and* $\{I_i\}$ *a collection of two-sided ideals. Suppose* S/I_i *is semisimple for all* i, *and that* $\bigcap_i I_i = (0)$. *Then* S *is semisimple.*

Proof. If S/I_i is semisimple, I_i is the intersection of the maximal left ideals that contain it. Thus $\bigcap_i I_i = (0)$ implies that the intersection of a certain collection of maximal left ideals is (0). This proves the lemma.

LEMMA 1.2. *Let* $\Omega = \{N_i\}$ *be a collection of normal subgroups of* G. *Suppose that for every finite subset* F *of* G *that does not contain* e, *there exists an* $N \in \Omega$ *that does not meet* F. *Let* R *be a ring, and suppose* R[G/N] *is semisimple for all* $N \in \Omega$. *Then* R[G] *is semisimple.*

Received December 23, 1965.

This paper was written while the author held an O.N.R. Research Associateship, ONR 432. The author thanks Professor Earl Lazerson for suggesting the problem under discussion.