THE FUNDAMENTAL EQUATIONS OF A SUBMERSION

Barrett O'Neill

1. INTRODUCTION

Let M and B be Riemannian manifolds. A *Riemannian submersion* π: M $\rightarrow$ B is a mapping of M onto B satisfying the following axioms, S1 and S2:

S1. π has maximal rank;

that is, each derivative map π_* of π is onto; hence, for each b $\in$ B, $\pi^{-1}(b)$ is a submanifold of M of dimension dim M - dim B. We use the language of fiber bundles, although π certainly need not be the projection of a bundle. In particular, the submanifolds $\pi^{-1}(b)$ are called *fibers*, and a vector field on M is *vertical* if it is always tangent to fibers, *horizontal* if always orthogonal to fibers; we use corresponding terminology for individual tangent vectors. The second axiom may now be stated in the following form.

S2. π_* preserves lengths of horizontal vectors.

Submersions occur widely in geometry (for example, as projection mappings of suitable Riemannian coset spaces). In classical geometry, a surface of revolution or a family of (so-called) parallel surfaces in R^3 each leads in an obvious way to a submersion. Further examples are given in Section 5, where in particular we compute (relative to a natural Riemannian structure) the sectional curvature of the frame bundle of a Riemannian manifold.

If we consider a submersion as the generalization of an isometry M $\rightarrow$ B to the case where dim M $\geq$ dim B, then the notion bears comparison with the generalization to dim M $\leq$ dim B, that is, with an isometric immersion. The character of an immersion is described by a single tensor, the second fundamental form. For a submersion we shall define two such tensors, one of which is the second fundamental form of all the fibers. Our purpose is to find the analogues, for a submersion, of the Gauss and Codazzi equations of an immersion, and thus, in particular, to find the relations linking the Riemannian curvatures of M, B, and the fibers $\pi^{-1}(b)$.

Certain aspects of submersions have been investigated, for example, by Hermann [1], and in greater generality ("bundle-like metrics") by Reinhart [4] and Hermann [2]. Our curvature results were suggested by the special case used by Kobayashi [3]. In preparing this paper we have benefited from conversations with A. Gray, who, in particular, suggested using the term "submersion" in this context.

2. THE FUNDAMENTAL TENSORS T AND A

For a submersion π: M $\rightarrow$ B, let $\mathscr{H}$ and $\mathscr{V}$ denote the projections of the tangent spaces of M onto the subspaces of horizontal and vertical vectors, respectively. (The same letters will serve for the horizontal and vertical distributions of Chevalley

Received April 9, 1966.
This work was supported by the National Science Foundation.