# A MILDLY WILD TWO-CELL

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## 1. INTRODUCTION

The results in this paper grew from an attempt to answer the following question of R. H. Fox: Does there exist in 3-space or in 4-space a wild 2-cell with an interior point p such that every 2-cell subset that has p on its boundary is tame? [5, Problem 21.] Doyle [4] has shown that no such cell exists in 3-space. In Section 5, we give an affirmative answer for 4-space, along with a discussion of mildly wild n-cells in (n + 2)-space. (An n-cell $C^n$ in $E^{n+2}$ is said to be *mildly wild* if it is wild and one of its interior points p has the property that each n-cell subset of $C^n$ having p on its boundary is tame.) In Section 3 we prove a general theorem on $\varepsilon$-taming. In Section 4 we prove an $\varepsilon$-taming theorem about almost piecewise linear imbeddings; it is the main tool in the construction of the mildly wild 2-cell; we also show, in Section 4, that each almost polyhedral 2-sphere in 4-space is the union of two flat cells.

## 2. DEFINITIONS

We assume familiarity with the material contained in Chapters 1 and 3 of [18], and we adhere to the notation given there. By a *simplex* we mean a closed rectilinear simplex, and by a *complex* we mean a closed rectilinear simplical complex (which may be assumed to be a subcomplex of a rectilinear division of some Euclidean space $E^q$). $K \downarrow L$ means that K *collapses* to L (see Chapter 3 of [7]). We shall abbreviate "piecewise linear" (or piecewise linearly) to pwl. If M is a manifold, we shall denote its *interior* by int M and its *boundary* by $\partial M$; we shall write $\text{Int}_X A$ for the interior of A as a subset of the topological space X, and $\bar{A}$ for the closure of A.

If a space C is homeomorphic (respectively, pwl homeomorphic) to a k-simplex, we say that C is a k-*cell* (respectively, k-*ball*). An n, m *cell pair* (n, m *ball pair*) is a pair $(C^n, C^m)$ of cells (balls) with $C^m \subset C^n$ and $C^m \cap \partial C^n = \partial C^m$; an n, m *semi-cell pair* (n, m *semi-ball pair*) is a pair $(C^n, C^m)$ of cells (balls) with $C^m \subset C^n$ and such that

$$C^m \cap \partial C^n = \partial C^m \cap \partial C^n = C^{m-1}$$

is an (m - 1)-cell (an (m - 1)-ball). The *standard* n, m *ball pair* $(\Sigma, \tau)$ and the *standard* n, m *semi-ball pair* $(\Sigma, \sigma)$ are defined as follows: let $\sigma'$ be an (m - 1)-simplex in $E^{m-1}$, and let $u = (0, \cdots, 0, -1)$ and $v = (0, \cdots, 0, 1)$ belong to $E^m$; then $\sigma$ is the m-simplex $u * \sigma$,

$$\tau = \sigma \cup (v * \sigma'),$$

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