A CLASS OF WEIGHT FUNCTIONS THAT ADMIT TCHEBYCHEFF QUADRATURE

J. L. Ullman

By a weight function $W(x)$ we mean a real-valued nonnegative function on $[-1, 1]$ for which the proper or improper Riemann integral exists and has the value 1. If the system of equations

$$(1) \qquad \frac{1}{n}\sum_{i=1}^{n} x_{i,n}^{k} = \int_{-1}^{1} x^k W(x)\,dx \qquad (k = 1, \cdots, n)$$

has real solutions for all positive integers n, we say that $W(x)$ *admits Tchebycheff quadrature.*

Hermite proved that the function $W(x) = 1/\pi\sqrt{1 - x^2}$ admits Techbycheff quadrature (see [1]). As far as the author knows, the literature lists no other examples.

THEOREM. *If* $-1/4 \leq a \leq 1/4$, *then the function*

$$(2) \qquad W(x) = \frac{1}{\pi\sqrt{1 - x^2}}\,\frac{1 + 2ax}{1 + 4a^2 + 4ax}$$

is a weight function and admits Tchebycheff quadrature.

This theorem establishes the existence of an infinite one-parameter family of what could properly be called Tchebycheff weight functions. It thus becomes reasonable to pose the problem of characterizing all Tchebycheff weight functions.

Proof of the theorem. In Lemmas 1 and 2 we develop a method for investigating the solutions of equations (1). We then apply this method to the weight function (2), in Lemmas 3 and 4, to complete the proof.

LEMMA 1. *Let* $W(x)$ *be a weight function, and let*

$$m_k = \int_{-1}^{1} x^k W(x)\,dx \qquad (k = 0, 1, \cdots).$$

The function

$$(3) \qquad f(z) = z \exp\left(-\sum_{k=1}^{\infty} \frac{m_k}{kz^k}\right) \qquad (|z| > 1)$$

has a simple pole at infinity, and $\lim_{z\to\infty} f(z)/z = 1$. *For each positive integer* n, *the terms with nonnegative powers of* z *in the Laurent expansion of* $(f(z))^n$ *about infinity form a monic polynomial* $F_n(z)$ *of degree* n.

To prove this lemma, we observe that $|m_k| \leq 1$ for all k, that the function defined by

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