NORMS OF POWERS OF ABSOLUTELY CONVERGENT FOURIER SERIES

G. W. Hedstrom

1. INTRODUCTION

For an absolutely convergent Fourier series

$$f(t) = \sum c_k e^{ikt}$$

we use the norm

$$\|f\| = \sum |c_k|,$$

and we seek estimates of $\|f^n\|$ as $n \to \infty$. Such estimates are important for the study of the behavior of the solution of the difference equation

$$(1.1) \qquad v_k^{n+1} = \sum_j c_j v_{k-j}^n \qquad (n = 1, 2, \cdots; k = 0, \pm 1, \cdots)$$

with preassigned values v_k^0 $(k = 0, \pm 1, \pm 2, \cdots)$. This difference equation is sometimes used to approximate a hyperbolic or parabolic partial differential equation (see [9], [11]). The solution of equation (1.1) can be written in the form

$$v_k^n = \sum_j c_{jn} v_{k-j}^0 \qquad (n = 1, 2, \cdots),$$

where c_{kn} is determined by the formula

$$f^n(t) = \sum_k c_{kn} e^{ikt} \qquad (n = 1, 2, \cdots).$$

Thus we have the inequality

$$\sup_k |v_k^n| \leq \|f^n\| \sup_k |v_k^0|,$$

and equality is attained if $v_k^0 = \exp\{i \arg c_{-k,n}\}$.

The behavior of $\|f^n\|$ as $n \to \infty$ has been the subject of several investigations. Beurling [7, pp. 428-429] proved that $\lim \|f^n\|^{1/n} = \max |f|$ (since f is necessarily continuous, $\max |f|$ exists). Beurling and Helson [1] and Leĭbenzon [6] proved that if $|f(t)| \equiv 1$ and $\|f^n\|$ is bounded, then $f(t) = \exp\{i(a_0 + kt)\}$ for some integer k and some real a_0. Also, it follows easily from their work that if $|f(t)| = 1$ on a set S of positive measure, $|f(t)| < 1$ on the complement of S, and $\|f^n\|$ is bounded, then $f(t) = \exp\{i(a_0 + kt)\}$, where a_0 is real and k is an integer. (The author is indebted

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