# SOME IN-BETWEEN THEOREMS FOR DARBOUX FUNCTIONS

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Suppose $\mathscr{C}$ is a class of real-valued functions on a real interval I, closed under scalar multiplication but not closed under addition. For f, g $\in \mathscr{C}$, we say that $g < f$ provided $g(x) < f(x)$ for all $x \in I$, and we raise the question whether the relation $g < f$ implies the existence of an element h in $\mathscr{C}$ such that $g < h < f$. Of course, if $\mathscr{C}$ were additive, then the average of f and g would give such an h.

We shall attack this question for the following specific classes: $\mathscr{D}$, the class of Darboux functions; $\mathscr{U}$, the uniform closure of $\mathscr{D}$; $\mathscr{D}\mathscr{B}_\alpha$, the class of Darboux functions that are Borel measurable of class $\alpha$; and $\mathscr{U}\mathscr{B}_\alpha$. All these classes have an interesting structure, yet they are badly nonadditive. For example, each function is the sum of two $\mathscr{D}$-functions; and $f + g \notin \mathscr{D}\mathscr{B}_1$ when f, g are the following $\mathscr{D}\mathscr{B}_1$-functions on the line:

$$f(x) = \sin\frac{1}{x} \text{ for } x \neq 0 \quad \text{and} \quad f(0) = 0,$$

$$g(x) = -f(x) \text{ for } x \neq 0 \quad \text{and} \quad g(0) = 1.$$

(For the foregoing facts about Darboux functions and for other information used in the sequel, we refer the reader to the expository paper [2] by Bruckner and Ceder.)

It turns out that there exists a pair of comparable $\mathscr{D}\mathscr{B}_2$-functions that admit no $\mathscr{D}$-function between them (Example 1). Nevertheless, we find a reasonable sufficient condition on a pair of comparable functions to admit a $\mathscr{D}$-function between them (Theorem 1), a condition that is satisfied, for example, by every pair of comparable $\mathscr{D}\mathscr{B}_1$-functions. Actually we prove that a $\mathscr{D}\mathscr{B}_2$-function can be inserted between two comparable $\mathscr{D}\mathscr{B}_1$-functions (Theorem 3). Whether this inserted function may be chosen to belong to $\mathscr{D}\mathscr{B}_1$ is an interesting unsolved problem. With reference to $\mathscr{U}$-functions, our results are more complete, in particular, we prove that any two comparable $\mathscr{U}$-functions admit an intermediate $\mathscr{U}$-function (Theorem 2).

Except where it is otherwise specified, all the functions considered in the sequel will be real-valued functions defined on some real interval I. For convenience, open and closed intervals will be denoted by (a, b) and [a, b], respectively, whether or not $a < b$. We think of an ordinal as the union of all smaller ordinals. Moreover, we shall consider cardinals as ordinals that are not equipollent with smaller ordinals. For any set A, $|A|$ denotes the cardinality of A. The cardinality of the reals is denoted by $\mathfrak{c}$. When $g < f$ and a, b $\in$ I, we define M(a, b) to be the open interval determined by $\min\{g(a), g(b)\}$ and $\max\{f(a), f(b)\}$.

A function f on I is called a *Darboux function* if it takes connected sets onto connected sets. In Bruckner, Ceder, and Weiss [3] the uniform closure $\mathscr{U}$ of $\mathscr{D}$ was characterized as the class of functions f such that $f([a, b] - C)$ is dense in $(f(a), f(b))$ whenever a, b $\in$ I with $f(a) \neq f(b)$ and $|C| < \mathfrak{c}$. Moreover, $\mathscr{U}\mathscr{B}_\alpha$ is precisely the uniform closure of $\mathscr{D}\mathscr{B}_\alpha$, as was shown in [3] and in Ceder and Weiss [5]. For facts about the function classes $\mathscr{B}_\alpha$, see Kuratowski [6].

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