ON THE MONOTONICITY OF THE ZEROS OF TWO POWER SERIES

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1. In the preceding paper, Peyerimhoff considers the functions f_κ and g_κ defined by the equations

$$f_\kappa(z) = \sum_{n=0}^{\infty} (n+1)^\kappa z^n \quad \text{and} \quad g_\kappa(z) = \sum_{n=0}^{\infty} (1 - c^{n+1})^\kappa z^n \quad (0 < c < 1)$$

in the unit circle, and by analytic continuation in C^*, the complex plane with a cut from 1 to ∞ along the positive real axis. In particular, he shows that these functions have exactly k zeros ($k < \kappa \leq k+1$) in C^*, and that the zeros are all negative and simple. His proof, as well as certain numerical calculations, indicate that the zeros are monotone functions of κ. For the sake of a better understanding of the functions f_κ and g_κ, it seems worthwhile to investigate this question. A proof that the two zeros of f_κ nearest to the origin and the first zero of g_κ are monotonic was communicated to me by A. Peyerimhoff. In this paper we shall show that *all zeros of* f_κ *and* g_κ *are monotonically increasing functions of* κ.

2. Let us consider f_κ first. We denote the zeros by $\xi_i(\kappa)$, with

$$0 > \xi_1(\kappa) > \cdots > \xi_k(\kappa).$$

From the paper of Peyerimhoff we take the relation

$$\xi_{i+1}(\kappa+1) < \xi_i(\kappa) < \xi_i(\kappa+1) \qquad (1 \leq i < \kappa). \tag{1}$$

Since the $\xi_i(\kappa)$ are simple and $f_\kappa(0) = 1 > 0$,

$$\operatorname{sgn} f'_\kappa(\xi_i(\kappa)) = (-1)^{i-1}.$$

From the relation

$$\frac{d\xi_i(\kappa)}{d\kappa} = -\frac{\partial f_\kappa(\xi_i(\kappa))}{d\kappa} \Big/ f'_\kappa(\xi_i(\kappa))$$

it will follow that $\frac{d}{d\kappa}\xi_i(\kappa) > 0$, when we have shown that

$$\operatorname{sgn} \frac{\partial f_\kappa(\xi_i(\kappa))}{d\kappa} = (-1)^i. \tag{2}$$

From the definition of f_κ we see that

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