THE SUM OF TWO CRUMPLED CUBES

Joseph Martin

A *crumpled cube* is a space that is homeomorphic to the closure of the interior of a 2-sphere in E^3. There exist many examples of crumpled cubes that are not 3-cells, the best known probably being the examples described by Alexander [1] and by Fox and Artin [7].

Suppose that C and D are crumpled cubes and h is a homeomorphism of Bd C onto Bd D. Let $C \cup_h D$ denote the space obtained by identification of C and D along their boundaries by the homeomorphism h.

Hosay [10] and Lininger [11] have independently shown that if C is a 3-cell, then $C \cup_h D$ is S^3. Bing has shown [4] that if each of C and D is the example of Alexander and h is the identity, then $C \cup_h D$ is S^3. It is known that if each of C and D is the example of Fox and Artin and h is the identity, then $C \cup_h D$ is not S^3.

The goal of this paper is to study certain conditions that are necessary in order that $C \cup_h D$ be S^3.

Suppose that K is a crumpled cube and p is a point of Bd K. The statement that p *is a piercing point of* K means that there exists an embedding $f: K \to S^3$ such that f(Bd K) can be pierced by a tame arc at f(p). If K is a 3-cell or the example of Alexander, then each point of Bd K is a piercing point of K. If K is the example of Fox and Artin, then K has exactly one nonpiercing point. Stallings [15] has given an example of a crumpled cube with uncountably many nonpiercing points.

The main result of this paper is the following theorem.

THEOREM. *Suppose that each of* C *and* D *is a crumpled cube,* h *is a homeomorphism of* Bd C *onto* Bd D, *and* $C \cup_h D$ *is* S^3. *Then, if* p *is a nonpiercing point of* C, h(p) *is a piercing point of* D.

We shall establish the theorem by using the theorem of Lininger and Hosay to view $C \cup_h D$ as a decomposition of S^3 into points and arcs. Lemma 6 will show that each arc in this decomposition is cellular, Lemma 4 will show that each of the arcs is locally tame except perhaps at *one* of its endpoints, and an application of Lemma 3 will establish the theorem.

LEMMA 1. *Suppose that* K *is a crumpled cube and* p *is a piercing point of* K. *Then, if* $f: K \to S^3$ *is an embedding such that* $Cl(S^3 - f(K))$ *is a* 3-*cell,* f(Bd K) *can be pierced by a tame arc at* f(p).

Proof. Let $f: K \to S^3$ be an embedding such that $Cl(S^3 - f(K))$ is a 3-cell, and let $g: K \to S^3$ be an embedding such that g(Bd K) can be pierced by a tame arc at g(p). It follows from a theorem of Gillman [8] that some tame arc α on g(Bd K) contains g(p).

The proof of Theorem 2 of [11] shows that there exists a homeomorphism $h: g(K) \to S^3$ such that (i) $Cl(S^3 - hg(K))$ is a 3-cell and (ii) the restriction of h to α is the identity. Now, since each of $Cl(S^3 - f(K))$ and $Cl(S^3 - hg(K))$ is a 3-cell, the homeomorphism hgf^{-1} can be extended to a homeomorphism of S^3 onto itself

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