THE EMBEDDINGS $O(n) \subset U(n)$ AND $U(n) \subset Sp(n)$, AND A SAMELSON PRODUCT

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In this paper we discuss the embeddings of the orthogonal group $O(n)$ in the unitary group $U(n)$ and of $U(n)$ in the symplectic group $Sp(n)$ induced by the embeddings $R \subset C$ and $C \subset H$, where R, C, and H are the fields of real numbers, complex numbers, and quaternions, respectively. We find that these monomorphisms are homotopic to a composite mapping

$$O(n) \xrightarrow{\phi} U(n-1) \xrightarrow{j} U(n),$$

where ϕ is an analytic function (but not a homomorphism) and j is the usual inclusion. A similar result holds for the embeddings $U(n) \to Sp(n)$, and we prove it simultaneously.

A more detailed study of the map $\phi: O(n) \to U(n-1)$ yields a further deformation $\tilde{\phi}: O(n) \to U(2[(n-1)/2])$, which is a "best possible" factorization (here [k] denotes the greatest integer in k). As we specialize further, the map $\phi: O(2n+1) \to U(2n)$ induces a map $\phi': V_{2n+1,2} \to S_{4n-1}$, which in turn induces the classical $\mathscr{C}_2$-isomorphism $\pi_k(V_{2n+1,2}) \approx \pi_k(S_{4n-1})$, where $\mathscr{C}_2$ is the class of 2-primary abelian groups. The map ϕ' can be made to yield some information on the 2-primary component of $\pi_k(V_{2n+1,2})$. We conjecture the existence of a similar map

$$O(2n+1) \to Sp(n)$$

that induces the $\mathscr{C}_2$-isomorphism $\pi_k(O(2n+1)) \approx \pi_k(Sp(n))$ of Harris [6], but work on this is incomplete. A more detailed study of the maps ϕ for the embeddings $U(n) \subset Sp(n)$ should yield results on this conjecture.

Finally, we use our results to calculate the order of the Samelson product $\langle \partial \iota_{2n}, \partial \iota_{2n} \rangle \in \pi_{4n-2}(O(2n))$, where ∂ is the transgression operator in the homotopy sequence of the fibration $O(2n+1) \to S_{2n}$. We are able to do this up to a factor of 2 for all n; and for $n \leq 4$, there is now sufficient knowledge of the homotopy groups of the appropriate Stiefel manifolds to calculate the exact order of this product. We are informed that by using entirely different methods, Mahowald [10] has calculated the order of $\langle \partial \iota_{2n}, \partial \iota_{2n} \rangle$ for $n \geq 4$.

1. NOTATION

Let F denote R, C, or H, where R is the field of real numbers, C is the field of complex numbers, and H is the field of quaternions. We use d for the dimension of F as an algebra over R, so that d = 1, 2, or 4. By F^n we denote an n-dimensional right vector space over F with a fixed basis and the usual inner product

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