MULTIPLICATIONS ON SO(3)

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1. INTRODUCTION

A multiplication on a space X (with base point $*$) will be defined to be a map $\mu: X \times X \to X$ such that $\mu(x, *) = \mu(*, x) = x$ for all x. Two multiplications μ_1 and μ_2 will be said to be homotopic if μ_1 is homotopic to μ_2 relative to $X \vee X$. The problem of enumerating the homotopy classes of multiplications that a given space may possess has been studied by James [3] for spheres, and by Arkowitz and Curjel [1] for finite CW-complexes. We shall prove the following theorem.

THEOREM 1.1. *There exist precisely* 768 *distinct homotopy classes of multiplications on* SO(3).

2. RESTATEMENT OF THE PROBLEM

SO(3) is homeomorphic to 3-dimensional real projective space P^3. We use K to denote the reduced product $P^3 \wedge P^3 = P^3 \times P^3/P^3 \vee P^3$. By [1], P^3 has as many multiplications as there are elements of $[K, P^3]$, the set of homotopy classes of base-point-preserving maps from K to P^3; since K is simply connected, the latter is clearly equivalent to $[K, S^3]$.

The space K has a standard CW-structure (see Section 5). If we write $K^{(n)}$ for the n-skeleton, then K is obtained from $K^{(5)}$ by attaching one 6-cell by means of a map of its boundary $S^5 \xrightarrow{h} K^{(5)}$. By [5], the following is an exact sequence of groups (we write Σ for suspension):

$$[S^5, S^3] \xleftarrow{h^*} [K^{(5)}, S^3] \longleftarrow [K, S^3] \longleftarrow [S^6, S^3] \xleftarrow{\Sigma h^*} [\Sigma K^{(5)}, S^3] \longleftarrow \cdots .$$

Since $[S^6, S^3] \simeq \pi_6(S^3) \simeq Z_{12}$, Theorem 1.1 is a consequence of the following three propositions.

PROPOSITION 2.1. $h^* = 0$.

PROPOSITION 2.2. $\Sigma h^* = 0$.

PROPOSITION 2.3. $[K^{(5)}, S^3]$ has order 2^6.

3. PROOFS OF PROPOSITIONS 2.1 AND 2.2

Proposition 2.1 asserts that gh is null-homotopic for each $g: K^{(5)} \to S^3$. Denote $K/K^{(2)}$ by L, and the natural projection $K \to L$ by p. Then the following implies 2.1.

PROPOSITION 3.1. *Let* $\tilde{g}: L^{(5)} \to S^3$ *be any map. Then* $\tilde{g}ph \sim *$.

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