EQUATIONS IN FREE METABELIAN GROUPS

Gilbert Baumslag and Kurt Mahler

Introduction. Equations in free groups have recently attracted considerable attention (see, for example, R. C. Lyndon and M. P. Schützenberger [3], G. Baumslag [1]). Free metabelian groups share many properties with free groups, and we now prove an analogue of a theorem about equations in free groups.

THEOREM. *If* a *and* b *are elements of a free metabelian group that are linearly independent modulo the derived group, and if* n *is any integer greater than* 1, *then* $a^n b^n$ *is not an* n*-th power.*

This theorem leaves unanswered a host of related questions. For example, if ℓ, m, and n are integers greater than 1, can $a^\ell b^m$ be an n-th power? This certainly seems unlikely. Of course, a and b must be linearly independent modulo the derived group; for if u and v are elements of a metabelian group and v lies in the derived group, then

$$(u^{-1})^2 (uv^2)^2 = (u^{-1} vu \cdot v)^2 .$$

We effect the proof of our theorem by first reducing it in a standard way to a problem in the group ring over the integers of a free abelian group (see G. Baumslag, Bernhard H. Neumann, Hanna Neumann, and Peter M. Neumann [2]) and then solving this problem with the help of elementary algebraic number theory.

The reduction to the group ring. Suppose that a and b are elements of a free metabelian group M and that they are linearly independent modulo M', the derived group of M. By a theorem of Nielsen [4] it follows that we can find an automorphism θ of M and a free set of generators x, y, z, ··· such that

$$a\theta \equiv x^\alpha \; (M'), \quad b\theta \equiv y^\beta \; (M') \qquad (\alpha > 0, \; \beta > 0).$$

We may therefore assume

$$a \equiv x^\alpha \; (M'), \quad b \equiv y^\beta \; (M') \qquad (\alpha > 0, \; \beta > 0). \tag{1}$$

The homomorphism η of M into M defined by

$$x\eta = x, \quad y\eta = y, \quad z\eta = 1, \quad \cdots$$

maps M into a free metabelian group of rank 2 in which $a\eta$ and $b\eta$ are themselves linearly independent modulo the derived group. Thus it suffices to settle the theorem for a free metabelian group M of rank 2 on x and y with a and b given by (1).

As usual, we put

Received March 23, 1965.

This paper was written at the Institute for Advanced Studies in Canberra. The first author gratefully acknowledges a grant from the National Science Foundation and the hospitality of the Australian National University.