A RADICAL FOR NEAR-RING MODULES

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The theory of various radicals for near-rings has been discussed by Betsch [1], Deskins [3], and Laxton [7]. It is our purpose here to study a radical for near-ring modules which, when restricted to near-rings with identity, coincides with the radicals defined by Betsch and Laxton.

In the second section we show that if $J(M)$ is the radical of a module M over a near-ring R, then $J(M/J(M)) = 0$. Also, if A is a submodule of M and $J(M/A) = 0$, then $J(M) \subseteq A$. These results were first obtained by Betsch and Laxton in the special case of a near-ring with identity.

In the third section we introduce the concepts of small and strictly small submodules. If the radical $J(M)$ of a near-ring module M is small (or strictly small), then $J(M)$ is the intersection of all maximal submodules (or maximal R-subgroups). Furthermore, $J(M)$ is the sum of all small submodules of M if and only if every submodule of M generated by a finite subset of $J(M)$ is small.

In the fourth section we restrict our attention to near-ring modules M that satisfy the descending chain condition on submodules. If the radical is the zero submodule, then M is a finite direct sum of minimal submodules. Let M be a finitely generated R-module. The radical $J(M)$ of M is small if and only if every maximal submodule of M is maximal as an R-subgroup.

1. FUNDAMENTAL DEFINITIONS

Definition 1. A *near-ring* R is a system with two binary compositions, addition and multiplication, such that

(i) the elements of R form a group R^+ under addition,

(ii) the elements form a semigroup under multiplication,

(iii) $x(y + z) = xy + xz$, for all $x, y, z \in R$,

(iv) $0 \cdot x = 0$, where 0 is the additive identity of R^+ and x is an element of R.

In particular, if R contains a multiplicative semigroup S whose elements generate R^+ and satisfy the condition

(v) $(x + y)s = xs + ys$ for all $x, y \in R$ and $s \in S$,

then R is called a *distributively generated* (d.g.) near-ring.

The most natural example of near-rings is given by the set of identity-preserving mappings of an additive group G (not necessarily abelian) into itself. If the mappings are added by adding images, and multiplication is iteration, then the system $(R, +, \cdot)$ is a near-ring. The near-ring R is called the near-ring associated with G.

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