ON THE MEAN VALUE OF NONNEGATIVE MULTIPLICATIVE NUMBER-THEORETICAL FUNCTIONS

Paul Erdős and A. Rényi

INTRODUCTION

A complex-valued function $g(n)$ $(n = 1, 2, 3, \cdots)$ defined on the set of natural numbers is called *multiplicative* if, for all pairs n, m of relatively prime natural numbers,

$$g(n \cdot m) = g(n) \cdot g(m). \tag{0.1}$$

A multiplicative function $g(n)$ is called *strongly multiplicative* if for all primes p and all positive integers k it satisfies the additional condition

$$g(p^k) = g(p),$$

and *completely multiplicative* if (0.1) holds for all pairs n, m of natural numbers. We say that the number-theoretical function $g(n)$ has a *mean value* if the limit

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} g(n) = M(g)$$

exists. The question of the existence of the mean value $M(g)$ has been much studied, but it has been solved only for certain subclasses. One of the definite results is the following theorem, due to H. Delange [2] (throughout the paper, p denotes a prime, and $\sum_p$ and $\prod_p$ denote a sum and a product, respectively, taken over all primes):

If $g(n)$ *is a strongly multiplicative number-theoretical function such that* $|g(n)| \leq 1$ *for* $n = 1, 2, \cdots$, *and such that the series*

$$\sum_p \frac{g(p) - 1}{p} \tag{0.2}$$

converges, then $M(g)$ *exists and*

$$M(g) = \prod_p \left(1 + \frac{g(p) - 1}{p}\right). \tag{0.3}$$

Conversely, if $g(n)$ *is a strongly multiplicative function such that* $|g(n)| \leq 1$ $(n = 1, 2, \cdots)$, $M(g)$ *exists, and* $M(g) \neq 0$, *then the series* (0.2) *converges,* $g(2) \neq -1$, *and* (0.3) *holds.*

In the present paper we shall consider only *real, nonnegative* multiplicative functions.

The following theorem has been proved by P. Erdős [3]:

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