AN EXTENSION OF LYAPUNOV'S DIRECT METHOD

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1. INTRODUCTION

Corduneanu [3] and Antosiewicz [1] observed that the direct method of Lyapunov depends basically on the fact that a function $m(t)$ satisfying the inequality $\dot{m}(t) \leq w(t, m(t))$ $(m(t_0) \leq r_0)$ is majorized by the maximal solution of the scalar differential equation $\dot{r} = w(t, r)$, $r(t_0) = r_0$. Lakshmikantham [4, 5] and others have made consistent use of this remark to extend the direct method of Lyapunov to various stability and boundedness problems.

Stability and boundedness are almost exclusively defined in terms of a distance from a given point. However, Ling [6] defined stability with respect to a manifold. In this paper we define stability and boundedness with respect to a manifold in a way more general than that of Ling. Then, by comparison with a scalar differential equation (as in previous papers of Lakshmikantham), we obtain theorems of boundedness and stability with respect to a manifold.

2. NOTATION AND DEFINITIONS

Let I denote the half-line $0 \leq t < +\infty$, and let R^n denote n-dimensional real euclidean space. We consider the system

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0 \qquad (t_0 \geq 0), \tag{2.1}$$

where x and f are n-vectors, where the function $f(t, x)$ is defined and continuous on the product space $I \times R^n$, and where $(\dot{\ }) = d/dt$.

Let g be a k-dimensional vector $(k \leq n)$, and suppose that the function $g(t, x)$ is defined and continuous on the product space $I \times R^n$. For each $t \in I$ let the set of points x satisfying the relation

$$g(t, x) = 0 \tag{2.2}$$

define an $(n - k)$-manifold $M_t(n - k)$.

Define $\|g(t, x)\|^2 = \sum_{i=1}^{k} g_i^2(t, x)$. For each $t \in I$ denote the sets

$$\{x: \|g(t, x)\| < \eta\} \qquad \text{and} \qquad \{x: \|g(t, x)\| \leq \eta\}$$

by $M_t(n - k)(\eta)$ and $M_t(n - k)(\bar{\eta})$, respectively. Suppose that $x(t)$ is any solution of the system (2.1).

In order to unify our results on stability and boundedness of the system (2.1) with respect to functions g satisfying (2.2), we list the following conditions:

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