LARGE AND SMALL SUBSPACES OF HILBERT SPACE

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In this paper we consider closed subspaces V of sequential Hilbert space ℓ_2 and of $L_2(0, 1)$. Our results are of two types: (1) if all the elements of V are "small," then V is finite-dimensional; (2) there exist infinite-dimensional subspaces V containing no small elements (except 0).

For example, Theorem 3 says that if V is a closed subspace of ℓ_2 and if $V \subset \ell_p$ for some $p < 2$, then V is finite-dimensional. On the other hand, the corollary to Theorem 4 states that there exist infinite-dimensional subspaces V of ℓ_2 none of whose nonzero elements belongs to any ℓ_p-space $(p < 2)$. [For $L_2(0, 1)$ the results are somewhat different: (1) if V is a closed subspace of $L_2(0, 1)$ and if $V \subset L_\infty$, then V is finite-dimensional. Theorem 6 gives a condition for the finite-dimensionality of V in terms of Orlicz spaces, and by Theorem 5 this condition is best possible; in particular, L_∞ cannot be replaced by L_q for any $q < \infty$. (2) There exist infinite-dimensional subspaces of L_2 none of whose nonzero elements is in any L_q-space $(q > 2)$ (Theorem 7)].

Since the elements $x \in \ell_2$ are functions $x = (x(1), x(2), \cdots)$ on the nonnegative integers, there are various ways of defining "small" elements. For example, Theorem 1 states that if all the elements $x \in V$ satisfy a condition $|x(n)| = O(\rho_n)$, where $\Sigma \rho_n^2 < \infty$, then V is finite-dimensional. On the other hand, Theorem 2 states that if $\Sigma \rho_n^2 = \infty$ then there exists an infinite-dimensional closed subspace V all of whose elements satisfy the condition $|x(n)| = O(\rho_n)$, but none of whose elements (except 0) satisfies the condition $|x(n)| = o(\rho_n)$.

Theorem 8 gives a formula for the exact dimension of any closed subspace V of ℓ_2. The paper concludes with an application of Theorem 8 to a problem involving bounded analytic functions in the unit disc: we give an elementary proof that an inner function cannot have a finite Dirichlet integral unless it is a finite Blaschke product.

We need the following compactness criterion [3, Chapter I, Section 10]:

If $\rho_n \geq 0$ and $\Sigma \rho_n^2 < \infty$, then $\{x: x \in \ell_2, \ |x(n)| \leq \rho_n\}$ is compact.

THEOREM 1. *Let* V *be a closed subspace of* ℓ_2, *and let* $\{\rho_n\}$ *be given, with* $\rho_n \geq 0$ *and* $\Sigma \rho_n^2 < \infty$. *If* $|x(n)| = O(\rho_n)$ *for all* $x \in V$, *then* V *is finite-dimensional.*

Proof. Let $V_m = \{x: x \in V, \ |x(n)| \leq m\rho_n \text{ for all } n\}$. Then V_m is compact and hence, if V were infinite-dimensional, V_m would be nowhere dense in V. But this would contradict the Baire category theorem, since $V = \bigcup V_m$.

THEOREM 2. *Let* $\rho_n > 0$, $\rho_n \to 0$ *and* $\Sigma \rho_n^2 = \infty$. *Then there exists an infinite-dimensional subspace* V *of* ℓ_2 *such that for each* $x \in V$

(i) $|x(n)| = O(\rho_n)$,

(ii) $|x(n)| = o(\rho_n) \Rightarrow x = 0$.

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