STIRLING SUMMABILITY OF RAPIDLY DIVERGENT SERIES

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1. INTRODUCTION

A summability method based on the Stirling numbers and a parameter λ was introduced by Karamata [5], who called it the Stirling method and denoted it by $S(\lambda)$. We shall use $\mathscr{S}(\lambda)$ for a slight modification of this method. The special case $\lambda = 1$ of $\mathscr{S}(\lambda)$ was studied independently by Lototsky [7] and developed by Agnew [1], [2], who named this case the Lototsky method, denoted by L. To illustrate the power of the method, Agnew showed that Euler's series $\Sigma(-1)^k k!\, z^{-k}$ is L-summable if (with $z = x + iy$) $x \geq \log 2$, but not if $|z| < \log 2$; the intermediate region remained in doubt [1, p. 111]. The purpose of the present note is to present a general theorem on Stirling summability, which will show in particular that Euler's series is L-summable if z is outside the first arch of $x = \log(2\cos y)$, but not if z is inside. By a separate argument, we can show that the series is summable on the boundary also. Furthermore, for $\mathscr{S}(\lambda)$-summability ($\lambda > 0$), we obtain the same region multiplied by λ; therefore the series is summable by some member of the family in the whole plane, except on the negative real axis.

It was pointed out by the referee that Greub [4] used the same curve $x = \log(2\cos y)$ for somewhat similar purposes. Greub's paper appeared almost simultaneously with [2], and it reached the same conclusions about the relations among the Lototsky and other summability methods.

2. DEFINITIONS

We define the Stirling numbers p_{nk} ($n = 1, 2, \cdots$; $k = 0, 1, 2, \cdots, n$) by the identity

$$x(x+1)(x+2)\cdots(x+n-1) = \sum_{k=0}^{n} p_{nk} x^k;$$

thus $p_{n0} = 0$ ($n = 1, 2, \cdots$), and we define also $p_{00} = 0$. The Stirling method was defined by Karamata by the formula

$$S(\lambda): \quad \sigma_n = \frac{1}{(\lambda)_n} \sum_{k=0}^{n} p_{nk} \lambda^k s_k,$$

where $(\lambda)_n = \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)$; if $\sigma_n \to \sigma$ as $n \to \infty$, we say the sequence $\{s_0, s_1, s_2, \cdots\}$ is $S(\lambda)$-limitable to σ. We always assume that $\lambda > 0$, which ensures regularity.

Received July 7, 1964.

This paper was written while the author was a member of the Summer Research Institute of the Canadian Mathematical Congress, Kingston, Ontario, 1964.