SOME RINGS WITH NIL COMMUTATOR IDEALS

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In a previous paper Drazin investigated certain conditions on a ring which would guarantee that its commutator ideal be nil [1]. In particular, he considered a class of rings, which he called K-rings, satisfying a certain Engel condition. However, as has been pointed out, the proof of one of the main theorems contains an error, and this in turn invalidates the proofs of several of the subsequent results [2], [3], [4]. In the present paper we shall prove one of these results and in addition obtain some related theorems. It should be mentioned that only a few of the proofs in [1] are invalid [2], [3]; indeed, in this paper we shall make use of several key results of [1], namely, Lemma 4.2 and Theorems 4.2 and 4.3.

We begin by recalling several classes of rings defined by Drazin in [1]. Let x and y be elements of a ring R. We define

$$e_0(x, y) = x, \qquad e_1(x, y) = [x, y] = xy - yx,$$

$$e_k(x, y) = [e_{k-1}(x, y), y] \qquad (k = 1, 2, \cdots).$$

Equivalently (proved easily by induction on k),

$$e_k(x, y) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} y^j x y^{k-j}.$$

If m is a positive integer, then a ring R is called an m-*ring* if and only if for every x, y in R, there exist integers k, t, n, and q and an element a in R such that $1 \leq t \leq m$ and

$$x^{m-t} e_k(qx^{t+1} + x^{t+1} a - x^t, y^n) = 0.$$

R is called a K-*ring* if and only if for every x, y in R, there exist integers $k = k(x, y)$ and $n = n(x, y)$ such that $e_k(x, y^n) = 0$. Thus every K-ring is a 1-ring, and if $t \leq m$, then every t-ring is an m-ring. Clearly, the properties of being a K-ring or an m-ring are preserved under homomorphism.

If R is any ring, the *Levitzki radical* of R, that is, the sum of all locally nilpotent ideals of R, will be denoted L(R) [9]. L(R) contains every locally nilpotent left (or right) ideal of R [5, p. 27]. The *Köthe radical* of R, that is, the sum of all nil ideals of R, will be denoted N(R) [8]. If J(R) denotes the Jacobson radical of R, then $L(R) \subset N(R) \subset J(R)$. It can easily be verified that the following assertions are equivalent for any ring R:

(i) the nilpotent elements of R form an ideal of R;

(ii) N(R) is precisely the set of nilpotent elements of R;

(iii) every nilpotent element of R generates a nil ideal of R;

(iv) R/N(R) has no nonzero nilpotent elements.

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