# A NOTE ON THE BORDISM ALGEBRA OF INVOLUTIONS

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## 1. INTRODUCTION

This note presents a structure theorem on the unoriented bordism group of involutions $\mathscr{R}_*(Z_2)$ defined in [1]. If we regard $\mathscr{R}_*(Z_2)$ as the bordism group of principal $Z_2$-bundles over closed manifolds, the tensor product of principal $Z_2$-bundles induces a multiplication in $\mathscr{R}_*(Z_2)$, making it an algebra over the Thom bordism algebra $\mathscr{R}_*$. On the other hand, we can also regard $\mathscr{R}_*(Z_2)$ as the singular bordism group $\mathscr{R}_*(B(Z_2))$ of a classifying space $B(Z_2)$. The diagonal map $\triangle: B(Z_2) \to B(Z_2) \times B(Z_2)$ then induces a comultiplication in $\mathscr{R}_*(Z_2)$, making it a coalgebra over $\mathscr{R}_*$. In summary, $\mathscr{R}_*(Z_2)$ becomes a Hopf algebra over $\mathscr{R}_*$. To study this Hopf algebra, we make use of the Smith homomorphism [1], whose existence is an additional special feature of $\mathscr{R}_*(Z_2)$. With all these structures on $\mathscr{R}_*(Z_2)$, we proceed to show that the Smith homomorphism helps to give some information on the comultiplication, which in turn yields some information on the multiplication. The information turns out to be just enough for a structure theorem. Our final conclusion states that $\mathscr{R}_*(Z_2)$ is an exterior algebra over $\mathscr{R}_*$ with generators in each dimension $2^n$ ($n = 0, 1, 2, \cdots$). As we shall see, this theorem is quite formal in nature, and it supplements in a modest way the work of P. E. Conner and E. E. Floyd. There are quite a few places where we are unable to be more explicit. The author is grateful to Professor Conner for many useful conversations.

## 2. GENERALITIES

We recall here the definition of the singular bordism group $\mathscr{R}_*(X)$ of a space X. We consider pairs $(M^n, f)$, where $M^n$ is a closed n-manifold and $f: M^n \to X$ is a continuous map. Two such pairs $(M_1^n, f_1)$ and $(M_2^n, f_2)$ are *bordant* if there exists a compact $(n+1)$-manifold $B^{n+1}$ and a map $F: B^{n+1} \to X$ such that the boundary of $B^{n+1}$ is the disjoint union of $M_1^n$ and $M_2^n$ and $F \mid M_i^n = f_i$ $(i = 1, 2)$. This is an equivalence relation, and the equivalence class of $(M, f)$ is denoted by $[M, f]$. (In [1], this class is denoted by $[M, f]_2$ to distinguish it from the oriented case.) The collection of all such classes is denoted by $\mathscr{R}_n(X)$, and $\mathscr{R}_*(X)$ is defined as $\Sigma_{n=0}^{\infty} \mathscr{R}_n(X)$. Disjoint union makes $\mathscr{R}_*(X)$ a vector space over $Z_2$. Moreover, $\mathscr{R}_*(X)$ is a module over the Thom unoriented bordism algebra $\mathscr{R}_*$. The module operation is given by $[M^n][N^m, f] = [M^n \times N^m, F]$, where F is the composition of projection onto $N^m$ followed by f. The module $\mathscr{R}_*(X)$ is also augmented. The augmentation $\varepsilon: \mathscr{R}_*(X) \to \mathscr{R}_*$ is given by $\varepsilon[M^n, f] = [M^n]$. The augmentation kernel is denoted by $\tilde{\mathscr{R}}_*(X)$, and it is called the *reduced module.*

Corresponding to the diagonal map $\triangle: X \to X \times X$ we have the induced homomorphism

$$\triangle_*: \mathscr{R}_*(X) \to \mathscr{R}_*(X \times X).$$

Received August 26, 1964.