AN INEQUALITY FOR THE DISCRIMINANT OF A POLYNOMIAL

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Let

$$f(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m = a_0 \prod_{h=1}^{m} (x - \alpha_h) \qquad (m \geq 2)$$

be an arbitrary polynomial with real or complex coefficients; put

$$L(f) = |a_0| + |a_1| + \cdots + |a_m|, \qquad M(f) = |a_0| \prod_{h=1}^{m} \max(1, |\alpha_h|).$$

Then, as I proved in [2],

$$(1) \qquad 2^{-m} L(f) \leq M(f) \leq L(f).$$

Here I shall establish and apply an upper estimate for the discriminant $D(f)$ of $f(x)$ in terms of either $L(f)$ or $M(f)$. This estimate is best-possible, and slightly better than one by R. Güting [1].

1. The main tool in the proof of the inequality is Hadamard's theorem on determinants, which may be stated as follows.

LEMMA 1. *If the elements of the determinant*

$$d = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

are arbitrary complex numbers, then

$$|d|^2 \leq \prod_{j=1}^{n} \left(\sum_{h=1}^{n} |a_{hj}|^2 \right),$$

and equality holds if and only if

$$\sum_{h=1}^{n} a_{hj} \bar{a}_{hk} = 0 \quad \textit{for } 1 \leq j < k \leq n.$$

Here $\bar{a}_{hk}$ denotes the complex conjugate of a_{hk}.

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