A REMARK ON MAXIMAL SUBRINGS

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A well-known theorem in group theory asserts that a finite group is solvable if it contains a maximal subgroup which is nilpotent and the Sylow 2-subgroup of which is sufficiently restricted (see [2], [5], [1], [3]). A similar "commutativity theorem" (without any finiteness conditions) holds for rings. It is the purpose of this note to prove the following proposition.

THEOREM. *If the maximal subring* M *of the ring* R *is solvable, then* M *is an ideal (containing all the additive commutators* ab - ba *of* R). *The set of all nilpotent elements of* R *is a solvable ideal; it is weakly nilpotent if* M *is weakly nilpotent.*

As in [4], we call an ideal I of the ring R *solvably (nilpotently) embedded* in R if for every homomorphism σ of R such that $I^\sigma \neq 0$ there is an ideal $J \neq 0$ of R^σ contained in I^σ such that $J^2 = 0$ ($R^\sigma J = JR^\sigma = 0$). The ring R is called *solvable (weakly nilpotent)* if it is a solvably (nilpotently) embedded ideal of itself.

Before proving the theorem we shall present our tools in a slightly more general form than is actually necessary. We shall make free use of propositions (S) and (N) of [4].

LEMMA 1. *Each solvable ideal* S *of the ring* R *is solvably embedded in* R.

Proof. By the general properties of the sum S(R) of all solvably embedded ideals of R (see Proposition (S) of [4]), we may assume that S(R) = 0. We shall now assume that the statement of the lemma is false, in other words, that $S \neq 0$, and then exhibit an ideal I of S with $I \neq I^2 = 0$. This contradiction yields the desired result. So let $A \neq 0$ be an ideal of S with $A^2 = 0$. Then clearly $(SA)^2 = 0$, and SA is a left ideal of R. Thus also the two-sided ideal SAR of R satisfies the equation $(SAR)^2 = 0$. Hence, if $SAR \neq 0$ we have arrived at the desired contradiction. If $SA \neq 0$ but SAR = 0, then SA is an ideal of R, and again we have a contradiction. But if SA = 0, then A is a left ideal of R, and hence $(AR)^2 = 0$. Thus either $AR \neq 0$ or A is an ideal of R; both cases yield the desired contradiction.

LEMMA 2. *If* N *is a weakly nilpotent ideal of the ring* R *such that* $R^2 \subseteq N$, *then the ring* R *is weakly nilpotent.*

Proof. By the general properties of the sum N(R) of all nilpotently embedded ideals of R (see Proposition (N) of [4]), we may assume that N(R) = 0, in other words, that no nonzero ideal in R annihilates R from both sides. We shall now assume that the statement of the lemma is false (that is, $R \neq 0$) and then exhibit an ideal of R that annihilates R from both sides. This contradiction yields the result. Since N(R) = 0, we see that $R^2 \neq 0$, hence $N \neq 0$. Let Z be the ideal of N consisting of all the elements of N that annihilate N from both sides; clearly Z is an ideal of R. If Z does not annihilate R from both sides, then $RZ \neq 0$, say, and

$$R(RZ) = (R)^2 Z \subseteq NZ = 0.$$

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