# GENERALISED CONSTANT WIDTH FOR MANIFOLDS

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## 1. INTRODUCTION

The notion of constant width may be formulated as follows. Let H be a compact $C^\infty$ n-manifold without boundary, convexly imbedded in real (n + 1)-space $R^{n+1}$. A chord of H is *normal* if it is normal to H at one of its two end-points, and *binormal* if it is normal to H at both end-points (compare Morse [3, p. 183]). Therefore H has *constant width* if every normal chord is binormal; for then all normal chords have the same length. Such a manifold is of course diffeomorphic to $S^n$. Conversely, any compact connected closed hypersurface in $R^{n+1}$ of constant width is convex and diffeomorphic to $S^n$.

In this paper we formulate more general conditions for manifolds imbedded with arbitrary codimension, and in some cases we obtain corresponding classification theorems.

Let V denote a smooth (that is $C^\infty$) connected n-manifold without boundary, smoothly imbedded in $R^m$ as a closed subset, for some $m > n$. We write $\nu(p)$ for the (m - n)-plane in $R^m$ normal to V at $p \in V$, and we say that V is *transnormal* in $R^m$ if, for each pair $p, q \in V$, the relation $q \in \nu(p)$ implies that $\nu(p) = \nu(q)$. Thus transnormality generalises constant width. It is easy to show that the map $\nu$ from a transnormal manifold V to the space of normal (m - n)-planes of V is a covering map. We say that V has *order* r or is r-*transnormal* if $\nu$ is r-fold. The main result is as follows.

THEOREM 1.1. *Any transnormal* n-*manifold of order* 2 *in* $R^m$ *is diffeomorphic to the cartesian product* $V_1 \times V_2$ *of differential manifolds* $V_1$, $V_2$, *where* $V_1$ *is homeomorphic to* $S^j$ *and* $V_2$ *is homeomorphic to* $R^{n-j}$ $(0 < j \leq n)$.

We do not know whether $V_1$ can have an unusual differential structure. For instance, can any of the 27 unusual 7-spheres be transnormally imbedded in $R^9$?

We show in Section 4 that for any transnormal manifold V and any $p, q \in V$, the sets $\nu(p) \cap V$ and $\nu(q) \cap V$ are isometric as subsets of $R^m$. We call $\nu(p) \cap V$ a *generating frame* of V, and we prove that the generating frame always admits a transitive group of isometries. This fact, together with Theorem 1.1, yields the following.

THEOREM 1.2. *If* V *is a transnormal* n-*manifold in* $R^{n+1}$ *of order* r, *then* $r = 2$ *or* $r = 1$.

Since it is easy to show that $R^n$ is (up to homeomorphisms) the only transnormal manifold of order 1, Theorems 1.1 and 1.2 classify transnormal hypersurfaces of finite order. In particular, the sphere, cylinder and plane are the only surfaces that can be transnormally imbedded in $R^3$ with finite order. The standard imbedding of the torus in $R^4$ is 4-transnormal.

The last statement is a consequence of the easily proved fact that if M and N can be transnormally imbedded in $R^a$, $R^b$ with orders $\lambda$ and $\mu$ respectively, then $M \times N$

Received July 8, 1963.
Supported by National Science Foundation grants GP-812 and G21938.